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Toward equivariant Iwasawa theory, II

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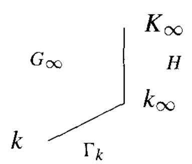
ABSTRACT

A "main conjecture" of equivariant Iwasawa theory in the non-Abelian situation is formulated, generalizing our work in [Ritter J., Weiss A., Manuscripta Math. **109** (2002) 131–146]. The new tool is a Hom description by means of which the l-adic Artin L-functions and the Iwasawa module X_∞ get related. The conjecture is confirmed in the so-called maximal order case.

In this paper we continue our work in [12], and, in particular, use the notation there (unless otherwise said). Recall the setup.



l is a given odd prime number and K_∞/k is a Galois extension of totally real number fields with $[k:\mathbb{Q}]$ finite and k_∞, the cyclotomic l-extension of k, contained in K_∞ with $[K_\infty : k_\infty] < \infty$. The respective Galois groups are $G_\infty = G_{K_\infty/k}$, $H = G_{K_\infty/k_\infty}$, $\Gamma_k = G_{k_\infty/k}$.

While [12] is mostly concerned with an equivariant "main conjecture" of Iwasawa theory, ($\Im$), for Abelian groups G_∞ we now no longer impose this restriction on G_∞ and formulate a non-commutative version of ($\Im$). This will be

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done by means of a Hom description which is to be regarded a substitute for $K_1(\mathcal{Q}(\mathbb{Z}_l[[G_\infty]]))$ in the localization sequence

$$K_1\big(\mathbb{Z}_l[[G_\infty]]\big) \to K_1\big(\mathcal{Q}(\mathbb{Z}_l[[G_\infty]])\big) \xrightarrow{\partial} K_0T\big(\mathbb{Z}_l[[G_\infty]]\big) \to K_0\big(\mathbb{Z}_l[[G_\infty]]\big).$$

The new ingredient is a homomorphism Det mapping $K_1(\mathcal{Q}(\mathbb{Z}_l[[G_\infty]]))$ to a certain Hom group $\mathrm{Hom}^*(R_l(G_\infty), (\mathbb{Q}_l{}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} \mathcal{Q}(\mathbb{Z}_l[[\Gamma_k]]))^\times)$, in which $R_l(G_\infty)$ denotes the additive group generated by all $\mathbb{Q}_l{}^{\mathrm{c}}$-valued characters χ of G_∞ with open kernel, where $\mathbb{Q}_l{}^{\mathrm{c}}$ is a fixed algebraic closure of $\mathbb{Q}_l$. In this Hom group we see a function L_k that incorporates the l-adic Artin L-functions for all such χ. Any generalization of $(\mathfrak{I})$ would have to specify the relation between this function L_k and the Iwasawa invariant $\mho_S$ (or rather a slight variant $\widetilde{\mho}_S$). Recall that $\mho_S$, which was introduced in [11,12] without assuming that G_∞ is Abelian, is a K-theory variant of the Iwasawa module $X_\infty = G_{M_\infty/K_\infty}$ with M_∞ the maximal Abelian l-extension of K_∞ unramified outside a given finite set S of primes of k containing the ones dividing l_∞. Actually, $\mho_S$ incorporates not only X_∞ (see Proposition 14) but is also strongly related to Chinburg's $\Omega(3)$-class [11].

The conjecture can now be stated as

there exists a unique element $\widetilde{\Theta}_S \in K_1(\mathcal{Q}(\mathbb{Z}_l[[G_\infty]]))$ *with* $\mathrm{Det}(\widetilde{\Theta}_S) = L_k$; *moreover,*

$$(\mathfrak{I}) \qquad \partial(\widetilde{\Theta}_S) = \widetilde{\mho}_S.$$

In the Abelian case this is [12, Theorem 11, p. 144] except for the $\mu = 0$ assumption.

In the present paper we prove that there is an $x \in K_1(\mathcal{Q}(\mathbb{Z}_l[[G_\infty]]))$ such that $\partial(x) = \widetilde{\mho}_S$ and that $\mathrm{Det}(x)L_k^{-1}$ is then a unit valued function, i.e., its value at every χ is in $(\mathbb{Z}_l{}^{\mathrm{c}} \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[[\Gamma_k]])^\times$. The full conjecture would require it to already belong to $\mathrm{Det}(K_1(\mathbb{Z}_l[[G_\infty]]))$.

In a forthcoming paper we shall extend the above result by reducing the conjecture (up to its uniqueness part) to $L_k \in \mathrm{Det}(K_1(\mathbb{Z}_l[[G_\infty]]_\wedge))$ for certain completions $_\wedge$ of $\mathbb{Z}_l[[G_\infty]]$.

For attempts at formulating a non-Abelian Iwasawa "main conjecture" in greater generality see, e.g., [5] and the references given there.

The paper is organized as follows. We start, in Section 1, with a collection of observations which will be convenient later. Then, in Section 2, we describe the algebra $\mathbb{Q}_l{}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} \mathcal{Q}(\mathbb{Z}_l[[G_\infty]])$ in terms of the $\mathbb{Q}_l{}^{\mathrm{c}}$-valued irreducible characters of G_∞ with open kernel (i.e., which factor through a finite quotient of G_∞). This enables us, in Section 3, to introduce the Hom description which is the main tool for formulating the above conjecture. In particular, we discuss $\mathrm{Det}\colon K_1(\mathcal{Q}(\mathbb{Z}_l[[G_\infty]])) \to \mathrm{Hom}^*(R_l(G_\infty), (\mathbb{Q}_l{}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} \mathcal{Q}(\mathbb{Z}_l[[\Gamma_k]]))^\times)$ and its relation to the reduced norm map on $\mathcal{Q}(\mathbb{Z}_l[[G_\infty]])$. The next section, Section 4, introduces $\widetilde{\mho}_S \in K_0T(\mathbb{Z}_l[[G_\infty]])$ and the function $L_k \in \mathrm{Hom}^*(R_l(G_\infty), (\mathbb{Q}_l{}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} \mathcal{Q}(\mathbb{Z}_l[[\Gamma_k]]))^\times)$. The conjecture then says how they are supposed to be connected.

We begin Section 5 by showing that $\widetilde{\mho}_S$ has a ∂-preimage x in $K_1(\mathcal{Q}(\mathbb{Z}_l[[G_\infty]]))$ and then compare the functions $\mathrm{Det}(x)$ and L_k at a given character χ of G_∞. The comparison depends on the classical main conjecture of Iwasawa theory, proved by Wiles [18], and is done via the $\mathfrak{o}[[\Gamma_k]]$-torsion module $\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} X_\infty)$, where M_χ is an integral realization of χ over some ring of integers $\mathfrak{o} \subset \mathbb{Q}_l{}^c$. This approach builds on the functorial version [17, Sections 6.1, 6.2] of the proof [4] of the l-adic Artin conjecture and, indeed, also yields its stronger form [4, p. 82], in Remark G. This results from defining an algebraic μ-invariant for all l-adic characters and then showing it agrees with the analytic one.

At an earlier stage in this investigation, David Burns and the second author discussed the formulation of “main conjectures” of this type. In particular, the possible validity of an equality analogous to $(\mathfrak{I})$ was raised, which, in turn, suggested the use of $\widetilde{\mho}$ in place of $\mho$.

1. PRELIMINARIES

We recall from [12] that $\mathcal{Q}(\mathbb{Z}_l[[G_\infty]])$ is the total ring of fractions of $\mathbb{Z}_l[[G_\infty]]$, obtained by adjoining inverses of all regular elements. If $\Gamma \simeq \mathbb{Z}_l$ is a central open subgroup of G_∞, then also

$$\mathcal{Q}(\mathbb{Z}_l[[G_\infty]]) = \left\{ \frac{a}{c} \colon a \in \mathbb{Z}_l[[G_\infty]],\ 0 \neq c \in \mathbb{Z}_l[[\Gamma]] \right\},$$

since the determinant of the action of a regular element on the free $\mathbb{Z}_l[[\Gamma]]$-module $\mathbb{Z}_l[[G_\infty]]$ is nonzero. Note that G_∞ splits over H. Namely, picking a preimage $x \in G_\infty$ of a topological generator γ_k of Γ_k we simply take $\gamma = (x^n)^{1/n}$, where n is the index of a Sylow pro-l subgroup L of G_∞; by [16, Exercise (1), p. 6], $x \mapsto x^n$ is bijective on L. In particular, a certain power of γ generates a central open Γ.

Lemma 1. *Let F be a finite field extension of $\mathbb{Q}_l$ with rings of integers $\mathfrak{o}$. Then*

$$F \otimes_{\mathbb{Q}_l} \mathcal{Q}(\mathbb{Z}_l[[G_\infty]]) = \mathcal{Q}(\mathfrak{o}[[G_\infty]]).$$

Proof. Take Γ as above and write the elements in $\mathcal{Q}(\mathfrak{o}[[G_\infty]])$ as a/b with $a \in \mathfrak{o}[[G_\infty]]$ and $0 \neq b \in \mathbb{Z}_l[[\Gamma]]$. Taking a $\mathbb{Z}_l$-basis $\{a_i\}$ of $\mathfrak{o}$, every element of $\mathcal{Q}(\mathfrak{o}[[G_\infty]])$ equals $\sum_i a_i x_i/b$ with $x_i \in \mathbb{Z}_l[[G_\infty]]$, hence it is the image of $\sum_i a_i \otimes x_i/b$ under the natural map $F \otimes_{\mathbb{Q}_l} \mathcal{Q}(\mathbb{Z}_l[[G_\infty]]) \to \mathcal{Q}(\mathfrak{o}[[G_\infty]])$. Conversely, if $\sum_i a_i \otimes x_i/b$ is in the kernel of this map, then multiplication by b yields $\sum_i a_i x_i = 0$, so $x_i = 0$. $\square$

Lemma 2. *Let χ be a $\mathbb{Q}_l{}^c$-valued character of G_∞ with open kernel and $F/\mathbb{Q}_l$ a finite field extension so that χ has a realization M_χ over the ring $\mathfrak{o}$ of integers of F. Moreover, let A be a finitely generated left $\mathbb{Z}_l[[G_\infty]]$-module. Then* $\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)$ *is a finitely generated* $\mathfrak{o}[[\Gamma_k]]$*-module such that* $(\gamma_k f)(m) = \gamma \cdot f(\gamma^{-1} m)$ *for* $\gamma \bmod H = \gamma_k$.

Proof. We show that the Γ_k-action above extends to an $\mathfrak{o}[[\Gamma_k]]$-action. Take a central Γ in the kernel of χ and let $\bar{\Gamma}$ be its image in Γ_k. Then $(\bar{\gamma} f)(m) = \gamma \cdot f(m)$ for $\gamma \in \Gamma$, $f \in \mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)$, $m \in M_\chi$. Since A is a $\mathbb{Z}_l[[G_\infty]]$-module, $\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)$ is an $\mathfrak{o}[[\bar{\Gamma}]]$-module, and finite generation follows. From $\mathfrak{o}[[\Gamma_k]] = \mathfrak{o}[[\bar{\Gamma}]][X]/(X^{[\Gamma_k:\bar{\Gamma}]} - \gamma_k^{[\Gamma_k:\bar{\Gamma}]})$ (with X an indeterminate) the lemma now results. □

Lemma 3. *Let γ_k be a topological generator of Γ_k and λ a prime element in $\mathfrak{o}$, where $\mathfrak{o}$ is as in Lemma 2. Moreover, for $x \in K_1(\mathcal{Q}(\mathfrak{o}[[\Gamma_k]]))$ let* $\det(x) = (\lambda^{m_1} f_1/\lambda^{m_2} f_2) \cdot u$ *be the Weierstraß decomposition relative to γ_k of* $\det(x)$ *in the field of fractions $\mathcal{Q}(\mathfrak{o}[[\Gamma_k]])$ of the Iwasawa algebra $\mathfrak{o}[[\Gamma_k]]$ (so $m_1, m_2 \geqslant 0$, $f_1, f_2 \in \mathfrak{o}[[\Gamma_k]]$ distinguished*[2] *and $u \in \mathfrak{o}[[\Gamma_k]]^\times$). The following holds.*

(1) *If $\partial : K_1(\mathcal{Q}(\mathfrak{o}[[\Gamma_k]])) \to K_0T(\mathfrak{o}[[\Gamma_k]])$ is the connecting homomorphism in the localization sequence for $\mathfrak{o}[[\Gamma_k]] \hookrightarrow \mathcal{Q}(\mathfrak{o}[[\Gamma_k]])$, then*

$$\partial(x) = \left[\mathfrak{o}[[\Gamma_k]]/\mathfrak{o}[[\Gamma_k]] \cdot \lambda^{m_1} f_1\right] - \left[\mathfrak{o}[[\Gamma_k]]/\mathfrak{o}[[\Gamma_k]] \cdot \lambda^{m_2} f_2\right].$$

(2) *The map $K_0T(\mathfrak{o}[[\Gamma_k]]) \xrightarrow{\text{char}} F[X]^\times$, induced by $M \mapsto$ characteristic polynomial of the action of $\gamma_k - 1$ on $F \otimes_{\mathfrak{o}} M$, satisfies* $(\mathrm{char}(\partial x))|_{X=\gamma_k-1} = f_1/f_2$.[3]
(3) *Let $\mathfrak{o}[[\Gamma_k]]_\bullet$ denote the localization of $\mathfrak{o}[[\Gamma_k]]$ with respect to the prime ideal generated by λ. The map $\mu_{\mathfrak{o}} : K_0T(\mathfrak{o}[[\Gamma_k]]) \to K_0T(\mathfrak{o}[[\Gamma_k]]_\bullet)$ takes $\partial(x)$ to $m_1 - m_2$ under the identification $K_0T(\mathfrak{o}[[\Gamma_k]]_\bullet) \simeq \mathbb{Z}$ given by $[\mathfrak{o}[[\Gamma_k]]_\bullet/\mathfrak{o}[[\Gamma_k]]_\bullet \cdot \lambda] \mapsto 1$.*

The lemma just recollects standard properties of the theory of modules over Iwasawa algebras and the relation to K_0T. Note that $\mathcal{Q}(\mathfrak{o}[[\Gamma_k]]) = \mathrm{Quot}(\mathfrak{o}[[\Gamma_k]]) = F \otimes_{\mathbb{Q}_l} \mathcal{Q}(\mathbb{Z}_l[[\Gamma_k]])$ by Lemma 1, and also that $\mathfrak{o}[[\Gamma_k]]_\bullet$ is a discrete valuation ring.

2. $\mathcal{Q}^c(G_\infty)$

We write $\Lambda(G_\infty)$ and $\mathcal{Q}(G_\infty)$ as abbreviations for $\mathbb{Z}_l[[G_\infty]]$ respectively its total ring of fractions $\mathcal{Q}(\mathbb{Z}_l[[G_\infty]])$. In this section we describe the algebra $\mathcal{Q}^c(G_\infty) \stackrel{\text{def}}{=} \mathbb{Q}_l{}^c \otimes_{\mathbb{Q}_l} \mathcal{Q}(G_\infty)$ in terms of the characters χ of irreducible $\mathbb{Q}_l{}^c$-representations of G_∞ with open kernel.

Lemma 4. *Let V_χ be a $\mathbb{Q}_l{}^c$-representation of G_∞ with irreducible character χ. Let η be an irreducible constituent of $\mathrm{res}^H_{G_\infty} \chi$ and set*

$$St(\eta) = \{g \in G_\infty \colon \eta^g = \eta\}, \quad e(\eta) = \frac{\eta(1)}{|H|} \sum_{h \in H} \eta(h^{-1}) h.$$

[2] Meaning that f_1, f_2 correspond to polynomials in T with leading terms 1 and all other coefficients divisible by λ, if $\mathfrak{o}[[\Gamma_k]]$ is identified with the ring $\mathfrak{o}[[T]]$ of power series by $\gamma_k - 1 \mapsto T$.

[3] Here, $F[X]$ is the polynomial ring in the indeterminate X over F and M is a torsion $\mathfrak{o}[[\Gamma_k]]$-module.

Then $e(\eta)V_\chi$ is a representation of $St(\eta)$ which is H-irreducible.

Proof. This follows from Clifford theory because $G_\infty/\ker\chi$ is finite and G_∞/H is procyclic. Indeed, $e(\eta)V_\chi$ is $St(\eta)$-irreducible and its restriction to H is $m\cdot\eta$ for some natural number m (compare [6, p. 565, 17.3 (e), (g)]). Now apply [6, p. 572, 17.12 (d)] to the character of the $St(\eta)$-module $e(\eta)V_\chi$. $\square$

Note that $St(\eta)$ does not depend on a special choice of $\eta \mid \mathrm{res}^H_{G_\infty}\chi$.

Proposition 5. (1) *$\mathcal{Q}^{\mathrm{c}}(G_\infty)$ is a semi-simple Artinian ring.*

(2) *Set $w_\chi = [G_\infty : St(\eta)]$ and $e_\chi = \sum_{\eta \mid \mathrm{res}^H_{G_\infty}\chi} e(\eta)$. Then e_χ is a central idempotent in $\mathcal{Q}^{\mathrm{c}}(G_\infty)$ and, given a generator γ_k of Γ_k, there exists a unique element*

$$\gamma_\chi \in \mathrm{cent}\big(\mathcal{Q}^{\mathrm{c}}(G_\infty)e_\chi\big)$$

so that

(i) *$\gamma_\chi = gc$ with $g \in G_\infty$ mapping to $\gamma_k^{w_\chi}$ modulo H and with $c \in (\mathbb{Q}_l{}^{\mathrm{c}}[H]e_\chi)^\times$,*
(ii) *γ_χ acts trivially on V_χ.*

Moreover, $\gamma_\chi = gc = cg$, γ_χ is central in $\mathcal{Q}^{\mathrm{c}}(G_\infty)e_\chi$, some l-power of γ_χ is in $G_\infty e_\chi$, and γ_χ generates a procyclic l-subgroup, Γ_χ, of $(\mathcal{Q}^{\mathrm{c}}(G_\infty)e_\chi)^\times$.

(3) *If ρ is a character of G_∞ of type W, i.e., $\mathrm{res}^H_{G_\infty}\rho = 1$, then $\gamma_{\chi\otimes\rho} = \gamma_\chi\rho(\gamma_k)^{-w_\chi}$.*

Proof. We prove the semi-simplicity of $\mathcal{Q}(G_\infty)$. From a splitting $G_\infty = H \rtimes \Gamma$, with $\Gamma = \langle\gamma\rangle \simeq \Gamma_k$, we get a presentation

$$\mathcal{Q}(G_\infty) = \bigoplus_{i=0}^{n-1} \mathcal{Q}(\Gamma^n)[H]\gamma^i$$

whenever Γ^n is central in G_∞. In particular, $\mathcal{Q}(G_\infty)$ is a finite-dimensional algebra over the field $\mathcal{Q}(\Gamma^n)$. Let now $\mathfrak{l}$ be a nilpotent left ideal in $\mathcal{Q}(G_\infty)$. Then multiplication by a suitable non-zero element of $\Lambda(\Gamma^n)$ $(= \mathbb{Z}_l[[\Gamma^n]])$ puts $\mathfrak{l}$ into $\Lambda(G_\infty)$. So we arrive at a nilpotent left ideal $\mathfrak{l}'$ in $\Lambda(G_\infty)$, whence $\mathfrak{l}' \bmod \gamma^{nj}-1$ is nilpotent in $\mathbb{Z}_l[G_\infty/\Gamma^{nj}] \subset \mathbb{Q}_l[G_\infty/\Gamma^{nj}]$ for all j. Therefore, $\mathfrak{l}'$, and so $\mathfrak{l}$, equals 0. This proves also (1).

Since $H \lhd G_\infty$, conjugation by elements $x \in G_\infty$ permutes the primitive central idempotents $e(\eta)$ of $\mathbb{Q}_l{}^{\mathrm{c}}[H]$ for $\eta \mid \mathrm{res}^H_{G_\infty}\chi$, hence e_χ is central.

We first prove the existence of γ_χ. Choose a $g \in G_\infty$ which maps to $\gamma_k^{w_\chi}$. Then $g \in St(\eta)$ for all $\eta \mid \mathrm{res}^H_{G_\infty}\chi$, and it acts on $V_\chi = \bigoplus_{\eta \mid \mathrm{res}^H_{G_\infty}\chi} e(\eta)V_\chi$ componentwise. Since, by Lemma 4, $e(\eta)V_\chi$ is H-irreducible, $g^{-1}|_{e(\eta)V_\chi} = c(\eta) \in \mathbb{Q}_l{}^{\mathrm{c}}[H]e(\eta) \simeq \mathrm{End}_{\mathbb{Q}_l{}^{\mathrm{c}}}(e(\eta)V_\chi)$. Now set $c = \sum_{\eta \mid \mathrm{res}^H_{G_\infty}\chi} c(\eta)$.

We turn to uniqueness. Let gc and g_1c_1 be two such elements, γ_χ. Then $g_1 = gh$ for some $h \in H$ and $(gc)^{-1}(g_1c_1) = c^{-1}hc_1 \overset{!}{=} e_\chi$. Indeed, $(gc)^{-1}(g_1c_1) = c^{-1}hc_1$ acts trivially on V_χ and belongs to $\mathbb{Q}_l{}^c[H]e_\chi$; however, the decomposition $V_\chi = \bigoplus_{\eta | \mathrm{res}^H_{G_\infty} \chi} e(\eta)V_\chi$ shows that the regular module $\mathbb{Q}_l{}^c[H]e_\chi$ is a submodule of some multiple of $\mathrm{res}^H_{G_\infty} V_\chi$.

To finish the proof of (2) we are left with showing

$gc = cg$, γ_χ is central in $\mathcal{Q}^c(G_\infty)e_\chi$, γ_χ generates a cyclic pro-l subgroup of $(\mathcal{Q}^c(G_\infty)e_\chi)^\times$, some power of γ_χ is in $G_\infty e_\chi$.

With respect to the first assertion we observe that $g^{-1}(gc)g = cg$ acts trivially on V_χ, so $(gc)(cg)^{-1} = (gcg^{-1})c^{-1} \in \mathbb{Q}_l{}^c[H]e_\chi$ acts trivially and thus equals e_χ, by the above argument.

For the second assertion we use the uniqueness of γ_χ, pick any $x \in G_\infty$ and consider $x\gamma_\chi x^{-1}$. Now $x\gamma_\chi x^{-1} = xgx^{-1} \cdot xcx^{-1}$ is just a second admissible decomposition of γ_χ, as $x\gamma_\chi x^{-1}$ acts trivially on V_χ.

Finally, for the third and fourth assertion we employ again the splitting $G_\infty = H \rtimes \Gamma$ and choose the group element g in Γ. Since χ has open kernel, a certain l-power of g acts trivially on V_χ, whence the same power of c acts trivially on V_χ and is thus equal to e_χ. At the same time this implies that some l-power of γ_χ is central in $G_\infty e_\chi$.

For the proof of (3) we first observe $e_{\chi\otimes\rho} = e_\chi$, $w_{\chi\otimes\rho} = w_\chi$, $V_{\chi\otimes\rho} = V_\chi \otimes_{\mathbb{Q}_l{}^c} V_\rho$. In particular we may write $\gamma_\chi = gc$ and $\gamma_{\chi\otimes\rho} = gc'$ with the same $g \in G_\infty$. Consequently, with $v \in V_\chi$ and $t \in V_\rho$,

$$\begin{aligned}\gamma_\chi(v \otimes t) &= gc(v \otimes t) \overset{1}{=} g(cv \otimes t) = gcv \otimes gt \overset{2}{=} \gamma_\chi v \otimes \gamma_k^{w_\chi} t \overset{3}{=} v \otimes \gamma_k^{w_\chi} t \\ &\overset{4}{=} \rho\big(\gamma_k^{w_\chi}\big)(v \otimes t).\end{aligned}$$

Here $\overset{1}{=}$ holds because $h \in H$ acts trivially on V_ρ and so, with $c = \sum_h \alpha_h h$, $c(v \otimes t) = \sum_h \alpha_h(h(v \otimes t)) = (\sum_h \alpha_h hv) \otimes t$, equation $\overset{2}{=}$ is due to $\gamma_\chi = gc$ and $\rho(g) = \rho(\gamma_k^{w_\chi})$, equation $\overset{3}{=}$ reflects the trivial action of γ_χ on V_χ and $\overset{4}{=}$ says that $\gamma_k^{w_\chi}$ acts via ρ on V_ρ. Recalling (ii) for $\gamma_{\chi\otimes\rho}$, we see that $\gamma_\chi\rho(\gamma_k^{-w_\chi}) = gc\rho(\gamma_k^{-w_\chi}) = \gamma_{\chi\otimes\rho}$ by $c\rho(\gamma_k^{-w_\chi}) \in (\mathbb{Q}_l{}^c[H]e_{\chi\otimes\rho})^\times$ and the uniqueness of $\gamma_{\chi\otimes\rho}$. $\square$

Notation. If F is a finite field extension of $\mathbb{Q}_l$ with ring of integers $\mathfrak{o}$, we set

$$\mathcal{Q}^F(G_\infty) = \mathcal{Q}\big(\mathfrak{o}[[G_\infty]]\big) \quad \text{and} \quad \Lambda^{\mathfrak{o}}(G_\infty) = \mathfrak{o}[[G_\infty]],$$

and similarly with G_∞ replaced by Γ_k or some Γ. Recall also Lemma 1 at this stage.

Proposition 6. *Let F be a finite field extension of $\mathbb{Q}_l$ with ring of integers $\mathfrak{o}$ so that the irreducible character χ has a realization V_χ over F. Then*

(1) $\Lambda^{\circ}(\Gamma_\chi)$, *with* Γ_χ *the closed group generated by* γ_χ, *has its field of fractions* $\mathcal{Q}^F(\Gamma_\chi)$ *contained in* $\mathcal{Q}^F(G_\infty)e_\chi$,

(2) $\gamma_\chi \in \mathrm{cent}(\mathcal{Q}^F(G_\infty)e_\chi)$ *induces* $\mathcal{Q}^F(\Gamma_\chi) \xrightarrow{\simeq} \mathrm{cent}(\mathcal{Q}^F(G_\infty)e_\chi)$,

(3) $\mathfrak{V}_\chi \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{F[H]}(V_\chi, \mathcal{Q}^F(G_\infty))$ *is an absolutely irreducible (right) module over* $\mathcal{Q}^F(\Gamma_k) \otimes_{\mathrm{cent}(\mathcal{Q}^F(G_\infty))} \mathcal{Q}^F(G_\infty)$, *where* $\mathrm{cent}(\mathcal{Q}^F(G_\infty))$ *acts on* $\mathcal{Q}^F(\Gamma_k)$ *by*

$$j_\chi : \mathrm{cent}\big(\mathcal{Q}^F(G_\infty)\big) \twoheadrightarrow \mathrm{cent}\big(\mathcal{Q}^F(G_\infty)e_\chi\big) \simeq \mathcal{Q}^F(\Gamma_\chi) \xrightarrow{\gamma_\chi \mapsto \gamma_k^{w_\chi}} \mathcal{Q}^F(\Gamma_k),$$

(4) j_χ *is independent of the choice of* γ_k.

Proof. Let $\Gamma(\simeq \mathbb{Z}_l)$ be a central subgroup of G_∞. Then $\mathcal{Q}^F(G_\infty) = \{a/c\colon\ a \in \Lambda^{\circ}(G_\infty),\ 0 \neq c \in \Lambda^{\circ}(\Gamma)\}$ and, by Proposition 5, there is a power N of l such that ${\gamma_\chi}^N \in \Gamma e_\chi$. From this (1) follows immediately.

We pass to (2). Since $e_\chi = \sum_{\eta \mid \mathrm{res}^H_{G_\infty} \chi} e(\eta)$, $F[H]e_\chi$ is the direct sum of matrix rings $F_{\eta(1)\times\eta(1)}$. By $\mathcal{Q}^F(\Gamma_\chi) \otimes_F F[H]e_\chi = \mathcal{Q}^F(\Gamma_\chi)[H]e_\chi$ and $\chi(1) = w_\chi \eta(1)$,

$$\dim_{\mathcal{Q}^F(\Gamma_\chi)}\big(\mathcal{Q}^F(\Gamma_\chi)[H]e_\chi\big) = \dim_F F[H]e_\chi = w_\chi \eta(1)^2.$$

On the other hand, writing $\gamma_\chi = gc$ as in Proposition 5, we see that $g = \gamma_\chi \otimes c^{-1} \in \mathcal{Q}^F(\Gamma_\chi)[H]e_\chi$ and consequently, on picking a $\gamma \in G_\infty$ so $\gamma \mapsto \gamma_k$, we obtain from $g \mapsto \gamma_k^{w_\chi}$

$$\mathcal{Q}^F(G_\infty)e_\chi = \sum_{i=0}^{w_\chi - 1} \mathcal{Q}^F(\Gamma_\chi)[H]e_\chi \cdot \gamma^i.$$

In particular, $\dim_{\mathcal{Q}^F(\Gamma_\chi)}(\mathcal{Q}^F(G_\infty)e_\chi) \leqslant w_\chi \cdot w_\chi \eta(1)^2 = \chi(1)^2$ and we have equality precisely when the sum is direct. This we check now.

Assume that $\sum_{i=0}^{w_\chi - 1} x_i \gamma^i = 0$ with

$$x_i \in \mathcal{Q}^F(\Gamma_\chi)[H]e_\chi = \sum_{\eta \mid \mathrm{res}^H_{G_\infty} \chi} \mathcal{Q}^F(\Gamma_\chi)[H]e(\eta).$$

Fixing one $\eta = \eta_0$, then all the η occurring in $\mathrm{res}^H_{G_\infty} \chi$ are the $\gamma^{-i}\eta_0\gamma^i = \eta_i$, $0 \leqslant i \leqslant w_\chi - 1$. On setting $e_i = e(\eta_i)$ and reading the indices i, j, r and differences of them in $\mathbb{Z}_l/\mathbb{Z}_l w_\chi$, we obtain

$$\begin{aligned} 0 = e_r \sum_i x_i \gamma^i e_j &= \sum_i e_r x_i e_{j-i} \gamma^i = e_r x_{j-r} \gamma^{j-r} \\ \Longrightarrow \quad e_r x_i = 0 \quad &\Longrightarrow \quad x_i = \sum_r e_r x_i = 0. \end{aligned}$$

At the same time (2) follows. Indeed, if $\sum_i x_i \gamma^i \in \mathrm{cent}(\mathcal{Q}^F(G_\infty)e_\chi)$, then $e_j \sum_i x_i \gamma^i = \sum_i x_i \gamma^i e_j = \sum_i x_i e_{j-i} \gamma^i$, so $e_j x_i = x_i e_{j-i} = e_j x_i e_{j-i}$. Since the e_i are central in $\mathcal{Q}^F(\Gamma_\chi)[H]e_\chi$, this last expression vanishes except for $i = 0$, whence $x_i = \sum_j e_j x_i = 0$ for $i \neq 0$. Therefore $x_0 \in \mathcal{Q}^F(\Gamma_\chi)[H]e_\chi$ is central, i.e., in $\sum_i \mathcal{Q}^F(\Gamma_\chi)e_i$, from which finally $x_0 \in \mathcal{Q}^F(\Gamma_\chi)e_\chi$ results, because $\gamma^{-1}x_0\gamma = x_0$.

It remains to verify (3) and (4). For (3) we first observe that $\mathfrak{V}_\chi$ is obviously a right $\mathcal{Q}^F(G_\infty)e_\chi$-module by $(fs)(v) = f(v)s$ (with $s \in \mathcal{Q}^F(G_\infty)e_\chi$, $f \in \mathfrak{V}_\chi$, $v \in V_\chi$); it is a left $\mathcal{Q}^F(\Gamma_k)$-module by Lemma 2. The actions of $1 \otimes \gamma_\chi$ and $\gamma_k^{w_\chi} \otimes 1$ agree where $\gamma_\chi = \gamma^{w_\chi} c$ as before. Then $(f \cdot (1 \otimes \gamma_\chi))(v) = (f\gamma_\chi)(v) = f(v)\gamma_\chi$ and

$$\begin{aligned}\big(f \cdot (\gamma_k^{w_\chi} \otimes 1)\big)(v) &= \big(\gamma_k^{w_\chi} f\big)(v) = \gamma^{w_\chi} f(\gamma^{-w_\chi} v) \stackrel{1}{=} \gamma^{w_\chi} c f(c^{-1}\gamma^{-w_\chi} v)\\ &\stackrel{2}{=} \gamma^{w_\chi} c f(v) = \gamma_\chi f(v) \stackrel{3}{=} f(v)\gamma_\chi.\end{aligned}$$

In these computations, $\stackrel{1}{=}$ is due to f being H-invariant, $\stackrel{2}{=}$ follows from the trivial action of $\gamma_\chi = \gamma^{w_\chi} c$ on V_χ, and $\stackrel{3}{=}$ from $\gamma_\chi \in \operatorname{cent} \mathcal{Q}^F(G_\infty)$.

After these preparations we conclude the proof of (3) of the proposition by showing

$$\dim_{\mathcal{Q}^F(\Gamma_k)} \mathfrak{V}_\chi = \chi(1).$$

This suffices, because the semi-simple algebra $\mathcal{Q}^F(G_\infty)e_\chi$ has a field as its center and so is simple, whence also $\mathfrak{A} \stackrel{\text{def}}{=} \mathcal{Q}^F(\Gamma_k) \otimes_{\mathcal{Q}^F(\Gamma_\chi)} \mathcal{Q}^F(G_\infty)e_\chi$ is simple. Now, the square of the dimension of any $\mathfrak{A}$-module is at least as big as $d \stackrel{\text{def}}{=} \dim_{\mathcal{Q}^F(\Gamma_k)} \mathfrak{A}$, and there exists one of dimension $\sqrt{d}$ if, and only if, $\mathfrak{A}$ splits. In our case, $d = \chi(1)^2$ and $\dim_{\mathcal{Q}^F(\Gamma_k)} \mathfrak{V}_\chi = \chi(1)$ because, using $[\mathcal{Q}^F(\Gamma_k) : \mathcal{Q}^F(\Gamma_\chi)] = w_\chi$, $\mathcal{Q}^F(G_\infty)e_\chi = \bigoplus_{i=0}^{w_\chi - 1} \mathcal{Q}^F(\Gamma_\chi)[H]e_\chi \cdot \gamma^i$ and $V_\chi = \bigoplus_{\eta | \operatorname{res}^H_{G_\infty} \chi} e(\eta)V_\chi$, we see that

$$\begin{aligned}w_\chi \dim_{\mathcal{Q}^F(\Gamma_k)} \mathfrak{V}_\chi &= \dim_{\mathcal{Q}^F(\Gamma_\chi)} \mathfrak{V}_\chi\\ &= \sum_{i=0}^{w_\chi - 1} \dim_{\mathcal{Q}^F(\Gamma_\chi)} \operatorname{Hom}_{F[H]}\big(V_\chi, \mathcal{Q}^F(\Gamma_\chi)[H]e_\chi \cdot \gamma^i\big)\\ &\stackrel{*}{=} w_\chi \cdot \dim_F \operatorname{Hom}_{F[H]}\big(V_\chi, F[H]e_\chi\big)\\ &= w_\chi \cdot \sum_\eta \dim_F \operatorname{Hom}_{F[H]}\big(e(\eta)V_\chi, F[H]e(\eta)\big)\\ &= w_\chi{}^2 \eta(1) = w_\chi \chi(1),\end{aligned}$$

with $\stackrel{*}{=}$ because Γ_χ acts trivially on $e(\eta)V_\chi$.

For (4) suppose that j_χ is defined by γ_k and j'_χ by $\gamma'_k = \gamma_k^v$ where $v \in \mathbb{Z}_l{}^\times$. Then $\gamma_\chi = gc$, $g \mapsto \gamma_k^{w_\chi}$ and $\gamma'_\chi = g'c'$, $g' \mapsto \gamma_k^{vw_\chi}$. Choosing $g' = g^v$ gives $\gamma'_\chi = \gamma_\chi^v$ by uniqueness. Hence $j'_\chi(\gamma'_\chi) = (\gamma'_k)^{w_\chi} = \gamma_k^{vw_\chi} = j_\chi(\gamma_\chi^v) = j_\chi(\gamma'_\chi)$ and $j'_\chi = j_\chi$. $\square$

Corollary. (1) *Each primitive central idempotent of $\mathcal{Q}^c(G_\infty)$ is an e_χ.*

(2) *Two primitive central idempotents e_χ, e_{χ_1} of $\mathcal{Q}^c(G_\infty)$ coincide if, and only if, $\chi_1 = \chi \otimes \rho$ for some character ρ of G_∞ of type W.*

Proof. To see (1), let e be a primitive central idempotent in $\mathcal{Q}^c(G_\infty)$. Then $e \cdot 1 = e$ and so $e\mathbb{Q}_l{}^c[H] \neq 0$. Hence there exists a $\mathbb{Q}_l{}^c$-irreducible character η_0 of H such that $e \cdot e(\eta_0) \neq 0$. Picking an irreducible constituent χ of $\mathrm{ind}_{H\times\Gamma}^{G_\infty}(\eta_0 \otimes 1)$, where Γ is central open in G_∞ with $\Gamma \cap H = 1$, then $\eta_0 \mid \mathrm{res}_{G_\infty}^{H} \chi$. Because e is central, $e \cdot e_\chi = e \sum_{\eta \mid \mathrm{res}_{G_\infty}^{H} \chi} e(\eta) \neq 0$ since the $e(\eta)$ are orthogonal idempotents. This implies $e = e_\chi$ since e_χ is also primitive by (2) of Proposition 6.

To see (2), apply (3) of Proposition 5 for $e_\chi = e_{\chi\otimes\rho}$. Conversely, if $e_\chi = e_{\chi_1}$, then $\mathrm{res}_{G_\infty}^{H} \chi = \mathrm{res}_{G_\infty}^{H} \chi_1$ and Clifford theory yields $\chi = \mathrm{ind}_{St(\eta)}^{G_\infty} \tilde{\eta}$, $\chi_1 = \mathrm{ind}_{St(\eta)}^{G_\infty}(\tilde{\eta} \otimes \kappa)$ with an extension $\tilde{\eta}$ to $St(\eta)$ of a fixed constituent η of $\mathrm{res}_{G_\infty}^{H} \chi$, and with a linear character κ of $St(\eta)/H \leqslant G_\infty/H = \Gamma_k$. Since the latter group is cyclic, κ can be extended to ρ on G_∞/H and then $\chi_1 = \chi \otimes \rho$. □

The corollary implies that $\mathcal{Q}^{\mathrm{c}}(G_\infty)$ splits into a direct product of simple rings $\mathcal{Q}^{\mathrm{c}}(G_\infty)e_\chi$ with χ running through a set of representatives of $\mathrm{Irr}_l(G_\infty)$ up to W-twists.

Definition. If ρ is a character of G_∞ of type W, then $\rho^\sharp$ is the automorphism of the field $\mathcal{Q}^{\mathrm{c}}(\Gamma_k)$ induced by $\rho^\sharp(\gamma_k) = \rho(\gamma_k)\gamma_k$ for $\gamma_k \in \Gamma_k$.

Note that in terms of the usual identification of $\Lambda(\Gamma_k)$ with power series over $\mathbb{Z}_l$, i.e., $\gamma_k - 1 \leftrightarrow T$ for a fixed generator γ_k of Γ_k, this $\rho^\sharp$ arises from the substitution $T \mapsto \rho(\gamma_k)T + (\rho(\gamma_k) - 1)$ on $\mathbb{Z}_l[\rho][[T]]$.

We denote by $\mathrm{Irr}_l(G_\infty)$ the set of all irreducible $\mathbb{Q}_l{}^{\mathrm{c}}$-valued characters χ of G_∞ with open kernel.

Theorem 7. *Let* $\mathrm{Maps}^{\mathrm{W}}(\mathrm{Irr}_l(G_\infty), \mathcal{Q}^{\mathrm{c}}(\Gamma_k))$ *denote the set of all maps*

$$f : \mathrm{Irr}_l(G_\infty) \to \mathcal{Q}^{\mathrm{c}}(\Gamma_k) \textit{ satisfying } f(\chi \otimes \rho) = \rho^\sharp\big(f(\chi)\big)$$

whenever ρ is of type W. Then there is a natural ring isomorphism between

$$\mathrm{cent}\big(\mathcal{Q}^{\mathrm{c}}(G_\infty)\big) \quad \textit{and} \quad \mathrm{Maps}^{\mathrm{W}}\big(\mathrm{Irr}_l(G_\infty), \mathcal{Q}^{\mathrm{c}}(\Gamma_k)\big).$$

Proof. Indeed, $z \in \mathrm{cent}(\mathcal{Q}^{\mathrm{c}}(G_\infty))$ induces the map f_z so $f_z(\chi) =$ image of ze_χ in $\mathcal{Q}^{\mathrm{c}}(\Gamma_k)$ under the map $j_\chi : \gamma_\chi \mapsto \gamma_k^{w_\chi}$. And if ρ is a character of G_∞ of type W, then $f_z(\chi \otimes \rho)$ is the image of $ze_{\chi\otimes\rho}$ in $\mathcal{Q}^{\mathrm{c}}(\Gamma_k)$, but now with respect to the map $j_{\chi\otimes\rho} : \gamma_{\chi\otimes\rho} \mapsto \gamma_k^{w_{\chi\otimes\rho}}$. By (3) of Proposition 5, $e_{\chi\otimes\rho} = e_\chi$, $w_{\chi\otimes\rho} = w_\chi$ and $\gamma_{\chi\otimes\rho} = \rho(\gamma_k)^{-w_\chi}\gamma_\chi$. This proves $f_z(\chi \otimes \rho) = \rho^\sharp(f_z(\chi))$.

Conversely, given f, then we must show that $f(\chi)$ is contained in the image of the centre field of $\mathcal{Q}^{\mathrm{c}}(G_\infty)e_\chi$ in $\mathcal{Q}^{\mathrm{c}}(\Gamma_k)$, so in $\mathcal{Q}^{\mathrm{c}}(\Gamma_k^{w_\chi})$ by (3) of Proposition 6. To see this, write $\chi = \mathrm{ind}_{St(\eta)}^{G_\infty}(\tilde{\eta})$ with $\tilde{\eta}$ an irreducible constituent in $\mathrm{res}_{G_\infty}^{St(\eta)} \chi$ (and $\eta \mid \mathrm{res}_{G_\infty}^{H} \chi$ as before). Then $\chi \otimes \rho = \chi$ for each irreducible character ρ of G_∞ whose kernel contains $St(\eta)$ and consequently $\rho^\sharp(f(\chi)) = f(\chi)$. Since these automorphisms $\rho^\sharp$ form the Galois group of $\mathcal{Q}^{\mathrm{c}}(\Gamma_k)/\mathcal{Q}^{\mathrm{c}}(\Gamma_k^{w_\chi})$, this implies the claim. □

Denote by nr the reduced norm map $\mathcal{Q}^{c}(G_\infty)^\times \to \mathrm{cent}(\mathcal{Q}^{c}(G_\infty))^\times$, so nr takes a unit $u \in \mathcal{Q}^{c}(G_\infty)$ to the central unit having component $\det(u|\mathfrak{V}_\chi)$ in $\mathrm{cent}(\mathcal{Q}^{c}(G_\infty)e_\chi)$, by (3) of Proposition 6. Of course, $\mathrm{nr}(\mathcal{Q}(G_\infty)) \subset \mathrm{cent}(\mathcal{Q}(G_\infty))$. Moreover, nr factors through $K_1(\mathcal{Q}^{c}(G_\infty))$, respectively through $K_1(\mathcal{Q}(G_\infty))$ (see [1, II, (40.31), p. 72]).

Given $\chi \in \mathrm{Irr}_l(G_\infty)$, we also define the map

$$\mathrm{Det}(\ )(\chi): K_1\big(\mathcal{Q}(G_\infty)\big) \to K_1\big(\mathcal{Q}^{c}(\Gamma_k)\big) \xrightarrow{\det} \mathcal{Q}^{c}(\Gamma_k)^\times$$

by

$$\begin{aligned}[P,\alpha] &\mapsto \big[\mathrm{Hom}_{\mathbb{Q}_l{}^c[H]}(V_\chi, \mathbb{Q}_l{}^c \otimes_{\mathbb{Q}_l} P), \alpha\big] \\ &\mapsto \det{}_{\mathcal{Q}^{c}(\Gamma_k)}\big(\alpha \mid \mathrm{Hom}_{\mathbb{Q}_l{}^c[H]}(V_\chi, \mathbb{Q}_l{}^c \otimes_{\mathbb{Q}_l} P)\big) = \mathrm{Det}\big([P,\alpha]\big)(\chi),\end{aligned}$$

where P is a projective $\mathcal{Q}(G_\infty)$-module and α a $\mathcal{Q}(G_\infty)$-automorphism of P.

Therefore we have the commutative triangles for all $\chi \in \mathrm{Irr}_l(G_\infty)$

$$\begin{tikzcd} & K_1(\mathcal{Q}(G_\infty)) \arrow[dl, "\mathrm{nr}"'] \arrow[dr, "\mathrm{Det}(\ )(\chi)"] & \\ \mathrm{cent}(\mathcal{Q}(G_\infty))^\times \arrow[rr, "j_\chi"] & & \mathcal{Q}^{c}(\Gamma_k)^\times \end{tikzcd}$$

by Proposition 6.

These triangles for all χ characterize the reduced norm because $\bigcap_\chi \ker j_\chi = 1$.

Theorem 8. *Let* $\mathrm{Hom}^*(R_l(G_\infty), \mathcal{Q}^{c}(\Gamma_k)^\times)$ *be the group of all homomorphisms* $f: R_l(G_\infty) \to \mathcal{Q}^{c}(\Gamma_k)^\times$ *satisfying*

$$\begin{cases} f(\chi \otimes \rho) = \rho^\sharp(f(\chi)) & \text{for characters } \rho \text{ of type } W, \\ f(\chi^\sigma) = f(\chi)^\sigma & \text{for all Galois automorphisms } \sigma \in G_{\mathbb{Q}_l{}^c/\mathbb{Q}_l}. \end{cases}$$

Then $x \mapsto [\chi \mapsto \mathrm{Det}(x)(\chi)]$ *defines a homomorphism*

$$\mathrm{Det}: K_1\big(\mathcal{Q}(G_\infty)\big) \to \mathrm{Hom}^*\big(R_l(G_\infty), \mathcal{Q}^{c}(\Gamma_k)^\times\big).$$

Proof. We start from Theorem 7, by which

$$\mathrm{cent}\big(\mathcal{Q}^{c}(G_\infty)\big) \simeq \mathrm{Maps}^{\mathrm{W}}\big(\mathrm{Irr}_l(G_\infty), \mathcal{Q}^{c}(\Gamma_k)\big).$$

Both sides carry a natural Galois action of $G_{\mathbb{Q}_l{}^c/\mathbb{Q}_l}$ and the isomorphism is Galois equivariant (see [1, II, pp. 329/330]). Taking fix points yields

$$(*) \qquad \mathrm{cent}\big(\mathcal{Q}(G_\infty)\big) \simeq \mathrm{Maps}^{\mathrm{W}}_{G_{\mathbb{Q}_l{}^c/\mathbb{Q}_l}}\big(\mathrm{Irr}_l(G_\infty), \mathcal{Q}^{c}(\Gamma_k)\big).$$

Since $R_l(G_\infty)$ is the free Abelian group on $\mathrm{Irr}_l(G_\infty)$, it follows that

$$\mathrm{cent}\big(\mathcal{Q}(G_\infty)\big)^\times \simeq \mathrm{Hom}^*\big(R_l(G_\infty), \mathcal{Q}^c(\Gamma_k)^\times\big),$$

so the triangles above finish the proof of the theorem. $\square$

As in Fröhlich's work on the Galois structure of rings of integers, the point of using a Hom description is, firstly, the replacement of the emphasis on the Wedderburn components of the algebra $\mathcal{Q}(G_\infty)$ by the functorial properties of characters, and, more importantly, the fact that the analytic invariants naturally appear in the Hom groups. For example,with respect to functoriality (compare [3, p. 62]), if G' is an open subgroup of G_∞ with fixed field k', respectively a factor group of G_∞ by a finite normal subgroup, and if $\chi' \in R_l(G')$, we define

$$\begin{aligned}
\mathrm{res}^{G'}_{G_\infty} &: \mathrm{Hom}^*\big(R_l(G_\infty), \mathcal{Q}^c(\Gamma_k)^\times\big) \to \mathrm{Hom}\big(R_l(G'), \mathcal{Q}^c(\Gamma_{k'})^\times\big),\\
\mathrm{defl}^{G'}_{G_\infty} &: \mathrm{Hom}^*\big(R_l(G_\infty), \mathcal{Q}^c(\Gamma_k)^\times\big) \to \mathrm{Hom}\big(R_l(G'), \mathcal{Q}^c(\Gamma_k)^\times\big)
\end{aligned}$$

by $(\mathrm{res}\, f)(\chi') = f(\mathrm{ind}^{G_\infty}_{G'}(\chi'))$ and $(\mathrm{defl} f)(\chi') = f(\mathrm{infl}^{G_\infty}_{G'}(\chi'))$.

Lemma 9. *Let θ stand for either* res *or* defl *and set accordingly $k_\theta = k'$ or k. Then*

(1) $\theta(\mathrm{Hom}^*(R_l(G_\infty), \mathcal{Q}^c(\Gamma_k)^\times)) \subset \mathrm{Hom}^*(R_l(G'), \mathcal{Q}^c(\Gamma_{k_\theta})^\times)$.
(2) *The restriction, respectively deflation, map θ on K_1 makes the diagram*

$$\begin{tikzcd}
K_1(\mathcal{Q}(G_\infty)) \arrow[r, "\theta"] \arrow[d, "\mathrm{Det}"'] & K_1(\mathcal{Q}(G')) \arrow[d, "\mathrm{Det}"'] \\
\mathrm{Hom}^*(R_l(G_\infty), \mathcal{Q}^c(\Gamma_k)^\times) \arrow[r, "\theta"] & \mathrm{Hom}^*(R_l(G'), \mathcal{Q}^c(\Gamma_{k_\theta})^\times)
\end{tikzcd}$$

commute.

Proof. We only prove the lemma for $\theta = \mathrm{res}$; the case $\theta = \mathrm{defl}$ is analogous (but easier, since the base field does not change).

For the first claim we need only to check whether restriction respects the type W property; compatibility with Galois automorphisms is obvious. This is done in the displayed computation below, in which f is in $\mathrm{Hom}^*(R_l(G_\infty), \mathcal{Q}^c(\Gamma_k)^\times)$ and ρ' a character of type W of G'. Viewing ρ' on $\Gamma_{k'}$, which group we regard as a subgroup of Γ_k, we see that ρ' is the restriction of a character ρ of G_∞ of type W.

$$\begin{aligned}
(\mathrm{res}\, f)(\chi' \otimes \rho') &= f\big(\mathrm{ind}(\chi' \otimes \rho')\big) = f\big(\mathrm{ind}(\chi') \otimes \rho\big)\\
&= \rho^\sharp\big(f\big(\mathrm{ind}(\chi')\big)\big) = \rho^\sharp\big((\mathrm{res}\, f)(\chi')\big) \overset{*}{=} (\rho')^\sharp\big((\mathrm{res}\, f)(\chi')\big).
\end{aligned}$$

For equation $\overset{*}{=}$ we first fix $\rho' = 1$ so that ρ varies over all characters of Γ_k which are trivial on $\Gamma_{k'}$, and obtain that $(\mathrm{res}\, f)(\chi') \in \mathcal{Q}^c(\Gamma_{k'})^\times$. But then $\overset{*}{=}$ follows from $\rho^\sharp|_{\mathcal{Q}^c(\Gamma_{k'})} = (\rho')^\sharp$.

We now turn to the diagram. Set $H' = G' \cap H$, $\Gamma' = \Gamma_{k'}$ (so $\Gamma' = G'/H'$), and let γ^i run through a set of representatives of Γ' in Γ_k. Typically, in what follows,

$g \in G_\infty$, $g' \in G'$, $h \in H$, $h' \in H'$, $v' \in V_{\chi'}$ (where $V_{\chi'}$ is a realization of χ' over $\mathbb{Q}_l{}^{\mathrm{c}}$), etc.

Claim. *Let P be a finitely generated $\mathcal{Q}(G_\infty)$-module. Then the map*

$$\begin{aligned}&\mathcal{Q}^{\mathrm{c}}(\Gamma_k) \otimes_{\mathcal{Q}^{\mathrm{c}}(\Gamma')} \mathrm{Hom}_{\mathbb{Q}_l{}^{\mathrm{c}}[H']}(V_{\chi'}, \mathbb{Q}_l{}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} P)\\ &\quad\to \mathrm{Hom}_{\mathbb{Q}_l{}^{\mathrm{c}}[H]}\big(\mathrm{ind}_{G'}^{G_\infty}(V_{\chi'}), \mathbb{Q}_l{}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} P\big),\end{aligned}$$

induced by

$$\gamma^i \otimes f' \mapsto \gamma^i f \quad \text{with } f(g \otimes v') = \begin{cases} hf'(g'v') & \text{if } g = hg' \in HG', \\ 0 & \text{if } g \notin HG', \end{cases}$$

is a $\mathcal{Q}^{\mathrm{c}}(\Gamma_k)$-vector space isomorphism.

Taking the Claim for granted for the moment, we proceed as follows. The commutativity of the diagram requires us to verify

$$\mathrm{res}_{G_\infty}^{G'}[P,\alpha] = [P',\alpha'] \implies \mathrm{Det}([P',\alpha'])(\chi') = \mathrm{Det}([P,\alpha])\big(\mathrm{ind}_{G'}^{G_\infty}(\chi')\big)$$

for pairs $[P,\alpha]$ consisting of a $\mathcal{Q}(G_\infty)$-module P and a $\mathcal{Q}(G_\infty)$-automorphism α of it. This, in turn, is equivalent to

$$\begin{aligned}&\det_{\mathcal{Q}^{\mathrm{c}}(\Gamma')}\big(\alpha' \mid \mathrm{Hom}_{\mathbb{Q}_l{}^{\mathrm{c}}[H']}(V_{\chi'}, \mathbb{Q}_l{}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} P)\big)\\ &\quad= \det_{\mathcal{Q}^{\mathrm{c}}(\Gamma_k)}\big(\alpha \mid \mathrm{Hom}_{\mathbb{Q}_l{}^{\mathrm{c}}[H]}\big(\mathrm{ind}_{G'}^{G_\infty}(V_{\chi'}), \mathbb{Q}_l{}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} P\big)\big),\end{aligned}$$

which follows from the naturality of the isomorphism of the Claim.

Regarding the Claim, one checks that the map

$$\begin{aligned}&\mathrm{Hom}_{\mathbb{Q}_l{}^{\mathrm{c}}[H]}\big(\mathrm{ind}_{G'}^{G_\infty}(V_{\chi'}), \mathbb{Q}_l{}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} P\big)\\ &\quad\to \mathcal{Q}^{\mathrm{c}}(\Gamma_k) \otimes_{\mathcal{Q}^{\mathrm{c}}(\Gamma')} \mathrm{Hom}_{\mathbb{Q}_l{}^{\mathrm{c}}[H']}(V_{\chi'}, \mathbb{Q}_l{}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} P)\end{aligned}$$

which sends f to $\sum_i \gamma^i \otimes f_i'$, where $f_i'(v') = (\gamma^{-i} f)(1 \otimes v')$, is inverse to that of the Claim. □

We finish this section by relating the K_1-map $\mathrm{Det}(\ )(\chi)$ to a K_0T-map on the next terms in the localization sequence. More explicitly, in the notation of Proposition 6, we study a map $K_0T(\Lambda(G_\infty)) \to K_0T(\Lambda^{\mathrm{o}}(\Gamma_k))$ induced by χ.

Proposition 10. *Let M_χ be an $\mathfrak{o}$-lattice on V_χ which is stable under the action of G_∞.*

(1) *If A is a finitely generated torsion $\Lambda(G_\infty)$-module with finite projective dimension, then* $\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)$ *is a finitely generated torsion $\Lambda^{\mathrm{o}}(\Gamma_k)$-module.*

(2) $A \mapsto [\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)]$ *induces a map* $K_0T(\Lambda(G_\infty)) \to K_0T(\Lambda^{\mathfrak{o}}(\Gamma_k))$ *which depends only on* χ, *not on* M_χ.

(3) *The diagram*

$$\begin{array}{ccc} K_1(\mathcal{Q}(G_\infty)) & \xrightarrow{\partial} & K_0T(\Lambda(G_\infty)) \\ \downarrow & & \downarrow \\ K_1(\mathcal{Q}^F(\Gamma_k)) & \xrightarrow{\partial} & K_0T(\Lambda^{\mathfrak{o}}(\Gamma_k)) \end{array}$$

commutes, with the right vertical map just defined and with the left vertical map $[P, \alpha] \mapsto [\mathrm{Hom}_{F[H]}(V_\chi, F \otimes_{\mathbb{Z}_l} P), \alpha]$.

Proof. By Lemma 2, $\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)$ is a finitely generated $\Lambda^{\mathfrak{o}}(\Gamma_k)$-module. And by its proof, any central $\Gamma \subset \ker \chi$ will have the $\Lambda(\overline{\Gamma})$-annihilator of A annihilating $\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)$. This settles (1).

For (2): If $A' \rightarrowtail A \twoheadrightarrow A''$ is a short exact sequence of finitely generated $\Lambda(G_\infty)$-torsion modules of finite projective dimension, then

$$\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} A') \rightarrowtail \mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} A) \twoheadrightarrow \mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} A'')$$

is exact because $H^1(H, \mathrm{Hom}_{\mathfrak{o}}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)) = 0$. In fact, A has finite projective dimension as $\mathfrak{o}[H]$-module (compare proof of [12, (2) of Proposition 4]), hence is cohomologically trivial [15, p. 152, Theorem 8] as required by [15, Theorem 9]. Therefore, each M_χ defines a map.

To show independence of the map from the choice of M_χ, it suffices to consider two lattices $M_\chi \subset M'_\chi$ such that $E = M'_\chi / M_\chi$ is annihilated by a prime element λ of $\mathfrak{o}$. We obtain the exact sequence

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{o}[H]}(E, \mathfrak{o} \otimes_{\mathbb{Z}_l} A) \rightarrowtail \mathrm{Hom}_{\mathfrak{o}[H]}(M'_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} A) &\to \mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} A) \\ &\twoheadrightarrow H^1\big(H, \mathrm{Hom}_{\mathfrak{o}}(E, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)\big), \end{aligned}$$

because (as above) $H^1(H, \mathrm{Hom}_{\mathfrak{o}}(M'_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)) = 0$. Since all $\Lambda^{\mathfrak{o}}(\Gamma_k)$-modules have finite projective dimension, we may read this sequence in $K_0T(\Lambda^{\mathfrak{o}}(\Gamma_k))$. Our claim will be established once we have verified that

$$\big[\mathrm{Hom}_{\mathfrak{o}[H]}(E, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)\big] = \big[H^1\big(H, \mathrm{Hom}_{\mathfrak{o}}(E, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)\big)\big].$$

Set $G = G_\infty / \ker \chi$ and $\overline{F} = \mathfrak{o}/\mathfrak{o}\lambda$. If C is an $\overline{F}[G]$-module, then writing $C = M_1/M_2$ with $\mathfrak{o}[G]$-lattices M_1, M_2 implies $H^r(H, \mathrm{Hom}_{\mathfrak{o}}(C, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)) = 0$ for $r \geqslant 2$ by the cohomological triviality of $\mathrm{Hom}_{\mathfrak{o}}(M_i, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)$ noted above. Hence, sending C to $[\mathrm{Hom}_{\mathfrak{o}[H]}(C, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)] - [H^1(H, \mathrm{Hom}_{\mathfrak{o}}(C, \mathfrak{o} \otimes_{\mathbb{Z}_l} A))]$ induces a map $G_0(\overline{F}[G]) \to K_0T(\Lambda^{\mathfrak{o}}(\Gamma_k))$ where $G_0(\overline{F}[G])$ is the Grothendieck group of $\overline{F}[G]$-modules. The surjectivity of the decomposition map $G_0(F[G]) \to G_0(\overline{F}[G])$ (see [1, I, p. 503, Theorem 21.16]) implies that $G_0(\overline{F}[G])$ is generated by the $[C]$ fitting into a short exact sequence $M \rightarrowtail M \twoheadrightarrow C$ with an $\mathfrak{o}[G]$-lattice M. We obtain

$$\mathrm{Hom}_{\mathfrak{o}[H]}(C, \mathfrak{o} \otimes_{\mathbb{Z}_l} A) \rightarrowtail \mathrm{Hom}_{\mathfrak{o}[H]}(M, \mathfrak{o} \otimes_{\mathbb{Z}_l} A) \to \mathrm{Hom}_{\mathfrak{o}[H]}(M, \mathfrak{o} \otimes_{\mathbb{Z}_l} A) \\ \twoheadrightarrow H^1\big(H, \mathrm{Hom}_{\mathfrak{o}}(C, \mathfrak{o} \otimes_{\mathbb{Z}_l} A)\big)$$

and our map takes $[C]$ to zero.

For (3): It is enough to check commutativity on the generators of $K_1(\mathcal{Q}(G_\infty))$ of the form $[\mathcal{Q}(G_\infty), \alpha]$ with $\alpha \in \Lambda(G_\infty)$ a unit in $\mathcal{Q}(G_\infty)$ (see [1, II, p. 76]). By definition of the maps, we have

$$\begin{tikzcd} {[\mathcal{Q}(G_\infty), \alpha]} \arrow[r, mapsto] \arrow[d, mapsto] & {[\Lambda(G_\infty)/\Lambda(G_\infty)\alpha]} \arrow[d, mapsto] \\ {[\mathrm{Hom}_{F[H]}(V_\chi, \mathcal{Q}^F(G_\infty)), \alpha]} & {[\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \Lambda^{\mathfrak{o}}(G_\infty)/\Lambda^{\mathfrak{o}}(G_\infty)\alpha)]} \end{tikzcd}$$

and we need to check that ∂ takes

$$\big[\mathrm{Hom}_{F[H]}\big(V_\chi, \mathcal{Q}^F(G_\infty)\big), \alpha\big] \quad \text{to} \quad \big[\mathrm{Hom}_{\mathfrak{o}[H]}\big(M_\chi, \Lambda^{\mathfrak{o}}(G_\infty)/\Lambda^{\mathfrak{o}}(G_\infty)\alpha\big)\big].$$

Since $\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \Lambda^{\mathfrak{o}}(G_\infty))$ is a $\Lambda^{\mathfrak{o}}(\Gamma_k)$-submodule of $\mathrm{Hom}_{F[H]}(V_\chi, \mathcal{Q}^F(G_\infty))$, which spans it as a $\mathcal{Q}^F(\Gamma)$-module (where, as before, $\Gamma \subset \ker\chi$), we see that

$$\partial\big[\mathrm{Hom}_{F[H]}\big(V_\chi, \mathcal{Q}^F(G_\infty)\big), \alpha\big] \\ = \mathrm{Hom}_{\mathfrak{o}[H]}\big(M_\chi, \Lambda^{\mathfrak{o}}(\Gamma_k)\big)/\mathrm{Hom}_{\mathfrak{o}[H]}\big(M_\chi, \Lambda^{\mathfrak{o}}(\Gamma_k)\big)\alpha.$$

The sequence $\Lambda(G_\infty) \overset{\cdot\alpha}{\rightarrowtail} \Lambda(G_\infty) \twoheadrightarrow \Lambda(G_\infty)/\Lambda(G_\infty)\alpha$ yields

$$\mathrm{Hom}_{\mathfrak{o}[H]}\big(M_\chi, \Lambda^{\mathfrak{o}}(G_\infty)\big) \overset{\cdot\alpha}{\rightarrowtail} \mathrm{Hom}_{\mathfrak{o}[H]}\big(M_\chi, \Lambda^{\mathfrak{o}}(G_\infty)\big) \\ \twoheadrightarrow \mathrm{Hom}_{\mathfrak{o}[H]}\big(M_\chi, \Lambda^{\mathfrak{o}}(G_\infty)/\Lambda^{\mathfrak{o}}(G_\infty)\alpha\big)$$

by $H^1(H, \mathrm{Hom}_{\mathfrak{o}}(M_\chi, \Lambda^{\mathfrak{o}}(G_\infty))) = 0$. $\square$

4. THE CONJECTURE

Let S be a finite set of primes of k containing all primes above ∞ and l and those whose ramification index (in K_∞) is divisible by l (compare [12, Remark, p. 135]). Then for each $\Lambda(G_\infty)$-monomorphism $\psi : \Lambda(G_\infty) \rightarrowtail \Delta G_\infty$ there is a commutative diagram (see [12, Section 1])

$$\begin{tikzcd} & \Lambda(G_\infty) \arrow[r, equal] \arrow[d, tail, "\Psi"'] & \Lambda(G_\infty) \arrow[d, tail, "\psi"'] \\ X_\infty \arrow[r, tail] \arrow[d, equal] & Y_\infty \arrow[r, two heads] \arrow[d] & \Delta G_\infty \arrow[d] \\ X_\infty \arrow[r, tail] & \mathrm{coker}\,\Psi \arrow[r, two heads] & \mathrm{coker}\,\psi \end{tikzcd}$$

in which the canonical middle row has Y_∞ of projective dimension $\leqslant 1$.

Since X_∞ and coker ψ are torsion $\Lambda(G_\infty)$-modules, we can define

$$\begin{aligned}\widetilde{\mho}_S &= [\operatorname{coker}\Psi] - \partial\big(\mathcal{Q}(G_\infty), \tilde{\psi}(1)\big)\\ &= [\operatorname{coker}\Psi] - [\operatorname{coker}\tilde{\psi}] \quad \text{in } K_0T\big(\Lambda(G_\infty)\big),\end{aligned}$$

where $\tilde{\psi}$ is ψ followed by the inclusion $\Delta G_\infty \subset \Lambda(G_\infty)$. By the same arguments as in [12, Section 1], $\widetilde{\mho}_S$ now depends only on S (but not on γ_k because, contrary to the ψ in [12, p. 135], $\tilde{\psi}$ no longer depends on γ_k).

By means of Brauer induction the l-adic Artin L-function $L_{l,S}(s,\chi)$ is defined for arbitrary characters χ of G_∞ with open kernel (see [4]). In the same way each topological generator γ_k of Γ_k permits the definition of $G_{\chi,S}(T) \in \mathbb{Q}_l{}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} \operatorname{Quot}(\mathbb{Z}_l[[T]])$ by starting out from the Deligne–Ribet power series $G_{\chi',S}(T)$ for Abelian characters χ' of open subgroups of G_∞ (see [2] or [10]). One then has

$$L_{l,S}(1-s,\chi) = \frac{G_{\chi,S}(u^s-1)}{H_\chi(u^s-1)}$$

where $u \in 1 + l\mathbb{Z}_l$ satisfies $\zeta^{\gamma_k} = \zeta^u$ for all l-power roots of unity, ζ, and where, for irreducible χ,[4]

$$H_\chi(T) = \begin{cases} \chi(\gamma_k)(1+T)-1 & \text{if } H \subset \ker\chi,\\ 1 & \text{else.}\end{cases}$$

We recall the basic properties of $G_{\chi,S}(T)$ and $H_\chi(T)$. Let $\phi_\chi(T)$ stand for either of them.

(1) $\phi_{\chi_1+\chi_2}(T) = \phi_{\chi_1}(T)\cdot\phi_{\chi_2}(T)$,
(2) $\phi_{\chi\otimes\rho}(T) = \phi_\chi(\rho(\gamma_k)(1+T)-1)$ if ρ has type W,[5]
(3) $\phi_{\mathrm{infl}_{G'}^{G_\infty}(\chi')}(T) = \phi_{\chi'}(T)$ if χ' is a character of $G' = G_\infty/N$ with $N \lhd G_\infty$ finite,
(4) $\phi_{\mathrm{ind}_{G'}^{G_\infty}(\chi')}(T) = \phi_{\chi'}((1+T)^{[k_\infty\cap k':k]}-1)$ if χ' is a character of an open subgroup G' of G_∞ with fixed field k',
(5) $\phi_{\chi^\sigma}(T) = \phi_\chi(T)^\sigma$ for $\sigma \in G_{\mathbb{Q}_l{}^{\mathrm{c}}/\mathbb{Q}_l}$.

These properties are a direct consequence of the analogous properties of the l-adic Artin L-function (compare [4] and also [17, Proposition 6.2.3]).

Definition. $L_k(\chi) = L_{k,S}(\chi) = \frac{G_{\chi,S}(\gamma_k-1)}{H_\chi(\gamma_k-1)}$.

Proposition 11. (1) $L_k \in \operatorname{Hom}^*(R_l(G_\infty), \mathcal{Q}^{\mathrm{c}}(\Gamma_k)^\times)$,
(2) *L_k is independent of the choice of γ_k.*

---

[4] Recall that $l \neq 2$. To cover $l = 2$ as well, it would be better to always set $H_\chi(T) = 1 - \chi(\gamma_k)(1+T)$ or $= 1$ according as $H \subset \ker\chi$ or not.

[5] Meaning irreducible with kernel containing H.

Proof. The first claim of the proposition is due to (1), (2), and (5) above. For the proof of (2) of the proposition let γ_k^v, with $v \in \mathbb{Z}_l^\times$, be a second generator of Γ_k. Then γ_k^v acts on the l-power roots of unity by u^v, so

$$L_{l,S}(1-s,\chi) = \frac{G_{\chi,S}(u^s-1)}{H_\chi(u^s-1)} = \frac{G_{\chi,S}^{(v)}(u^{vs}-1)}{H^{(v)}(u^{vs}-1)}$$

with the exponent $^{(v)}$ referring to the generator γ_k^v. By Weierstraß preparation it follows that

$$\frac{G_{\chi,S}(T)}{H_\chi(T)} = \frac{G_{\chi,S}^{(v)}((1+T)^v-1)}{H^{(v)}((1+T)^v-1)},$$

since clearing denominators yields power series in $\mathbb{Q}_l{}^{\mathrm{c}} \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[[T]]$ having the same values at $T = u^s - 1$ for $s \in \mathbb{Z}_l$. Now substitute $T = \gamma_k - 1$. $\square$

The proposition and Theorem 8 allow us to state the conjecture which is already in the introduction.

Conjecture. *There is a unique element* $\widetilde{\Theta}_S \in K_1(\mathcal{Q}(G_\infty))$ *with* $\mathrm{Det}(\widetilde{\Theta}_S) = L_{k,S}$. *Moreover,* $\partial(\widetilde{\Theta}_S) = \widetilde{\mho}_S$.

Remarks. (A) This conjecture generalizes that of [12] to the non-Abelian case because $\widetilde{\Theta}_S, \widetilde{\mho}_S$ are related to $\Theta_S, \mho_S$ as follows.

Let $\gamma \mapsto \gamma_k$ induce a splitting of G_∞ over H and set $e = e(1_H) = \frac{1}{|H|}\sum_{h\in H} h$. Then

$$\mathrm{Det}\big[\mathcal{Q}(G_\infty), (\gamma-1)e + (1-e)\big](\chi) = H_\chi(\gamma_k - 1).$$

This is because for irreducible χ (and $f \in \mathrm{Hom}_{\mathbb{Q}_l{}^{\mathrm{c}}[H]}(V_\chi, \mathcal{Q}^{\mathrm{c}}(\Gamma_k))$)

$$\begin{aligned}
&\mathrm{Det}\big[\mathcal{Q}(G_\infty), (\gamma-1)e + (1-e)\big](\chi)\\
&\quad = \det_{\mathcal{Q}^{\mathrm{c}}(\Gamma_k)}\big((\gamma-1)e + (1-e) \mid \mathrm{Hom}_{\mathbb{Q}_l{}^{\mathrm{c}}[H]}\big(V_\chi, \mathcal{Q}^{\mathrm{c}}(G_\infty)\big)\big)\\
&\quad = \det_{\mathcal{Q}^{\mathrm{c}}(\Gamma_k)}\big(\gamma_k - 1 \mid \mathrm{Hom}_{\mathbb{Q}_l{}^{\mathrm{c}}[H]}\big(V_\chi, \mathcal{Q}^{\mathrm{c}}(\Gamma_k)\big)\big)\\
&\quad = \begin{cases} 1 & \text{if } \chi \text{ is not of type W},\\ H_\chi(\gamma_k - 1) & \text{if } \chi \text{ is of type W, by } f\cdot(\gamma_k-1) = (\chi(\gamma_k)\gamma_k - 1)\cdot f. \end{cases}
\end{aligned}$$

Now, the existence of $\widetilde{\Theta}_S$ is equivalent to the existence of $\Theta_S \in K_1(\mathcal{Q}(G_\infty))$ such that $\mathrm{Det}(\Theta_S)(\chi) = G_{\chi,S}(\gamma_k - 1)$. In particular, the class $\mho_S$ of [12, p. 135] clearly satisfies $\mho_S = \widetilde{\mho}_S + \partial(\mathcal{Q}(G_\infty), (\gamma-1)e + (1-e))$. So we have $[\partial(\widetilde{\Theta}_S) = \widetilde{\mho}_S \iff \partial(\Theta_S) = \mho_S]$.

(B) D. Burns has informed us of a paper in progress, *Non-Abelian Iwasawa theory for Tate motives*. This paper discusses an analogous conjecture in the language of étale cohomology and avoiding the use of Det. This raises the possibility of formulating "main conjectures" for more general G_∞.

(C) On the other hand, if Leopoldt's conjecture is true, $\mho_S \in K_0R(\Lambda(G_\infty))$ where R is the multiplicative set of "never zero divisors" (see [11, p. 31]). This has the advantage that $\mho_S$ deflates to finite level.

(D) When G_∞ is Abelian, then the reduced norm is an isomorphism (see [12, Lemma 5a]) and the Conjecture is equivalent with ($\mathfrak{I}$) in [12].

(E) The triangles prior to Theorem 8 and the isomorphism

$$\mathrm{cent}\big(\mathcal{Q}(G_\infty)\big)^\times \simeq \mathrm{Hom}^*\big(R_l(G_\infty), \mathcal{Q}^c(\Gamma_k)^\times\big)$$

imply that Det is surjective, respectively injective, precisely when the reduced norm $\mathrm{nr}\colon K_1(\mathcal{Q}(G_\infty)) \to \mathrm{cent}(\mathcal{Q}(G_\infty))^\times$ is. However, in general nr is neither (see, e.g., [9]). The uniqueness statement in the Conjecture is equivalent to $\ker(\mathrm{nr}) = SK_1(\mathcal{Q}(G_\infty)) = 1$. Suslin (see [7, p. 250]) conjectures that $SK_1(\mathfrak{A}) = 1$ for central simple algebras $\mathfrak{A}$ over fields with cohomological dimension $\leqslant 3$. The fields $\mathcal{Q}(\Gamma)$, with $\Gamma \simeq \mathbb{Z}_l$, have cohomological dimension 3 by a result of Kato (see [14]),[6] hence the centre fields in $\mathrm{cent}(\mathcal{Q}(G_\infty))$ enjoy the same property (see [8, p. 315]).

5. EVIDENCE IN FAVOUR OF THE CONJECTURE

Proposition 12. (1) *Let $G' = G_\infty/N$ with a finite normal subgroup N of G_∞, and put $K'_\infty = K_\infty^N$. Then*

(a) $\mathrm{defl}\colon K_0T(\Lambda(G_\infty)) \to K_0T(\Lambda(G'))$ *takes* $\widetilde{\mho}_S$ *to* $\widetilde{\mho}'_S = \widetilde{\mho_S(K'_\infty/k)}$.

(b) $\mathrm{defl}\colon \mathrm{Hom}^*(R_l(G_\infty), \mathcal{Q}^c(\Gamma_k)^\times) \to \mathrm{Hom}^*(R_l(G'), \mathcal{Q}^c(\Gamma_k)^\times)$ *takes* $L_k = L_{K_\infty/k}$ *to* $L'_k = L_{K'/k}$.

(2) *Let G' be an open subgroup of G_∞, and put $k' = K_\infty^{G'}$. Then*

(a) $\mathrm{res}\colon K_0T(\Lambda(G_\infty)) \to K_0T(\Lambda(G'))$ *takes* $\widetilde{\mho}_S$ *to* $\widetilde{\mho}'_S = \widetilde{\mho_S(K_\infty/k')}$.

(b) $\mathrm{res}\colon \mathrm{Hom}^*(R_l(G_\infty), \mathcal{Q}^c(\Gamma_k)^\times) \to \mathrm{Hom}^*(R_l(G'), \mathcal{Q}^c(\Gamma_{k'})^\times)$ *takes* $L_k = L_{K_\infty/k}$ *to* $L_{k'} = L_{K_\infty/k'}$.

Proof. Assertion (1)(a) is [11, Proposition 4.8]. Assertions (1)(b) and (2)(b) follow from properties (3), respectively (4), of the previous section with ϕ replaced by L_k. Assertion (2)(a) follows from the proof of Lemma A.2(1) in [13] or, more directly, from assembling the three diagrams [13, p. 273] into the commutative cube

[6] We would like to thank K. Kato for this reference.

$$\begin{tikzcd}
& \Lambda(G'_\infty) \arrow[rr, equal] \arrow[dl, "\Psi'"'] \arrow[dd, tail] & & \Lambda(G'_\infty) \arrow[dl, "\tilde{\psi}'"'] \arrow[dd, tail] \\
Y'_\infty \arrow[rr] \arrow[dd, tail] & & \Lambda(G'_\infty) \arrow[dd, tail] & \\
& \Lambda(G_\infty) \arrow[rr, equal] \arrow[dl, "\Psi"'] \arrow[dd] & & \Lambda(G_\infty) \arrow[dl, "\tilde{\psi}"'] \arrow[dd] \\
Y_\infty \arrow[rr] \arrow[dd] & & \Lambda(G_\infty) \arrow[dd] & \\
& F \arrow[rr, equal] \arrow[dl, "D"'] & & F \arrow[dl, "d"'] \\
F \arrow[rr, "\simeq"] & & F &
\end{tikzcd}$$

by setting $\tilde{\psi}(1) = \tilde{\psi}'(1)$, $\Psi(1) = \Psi'(1)$. All southwest arrows are injective. From the cube we obtain

$$\begin{tikzcd}
\operatorname{coker} \Psi' \arrow[d, tail] & \operatorname{coker} \tilde{\psi}' \arrow[d, tail] \\
\operatorname{coker} \Psi \arrow[d] & \operatorname{coker} \tilde{\psi} \arrow[d] \\
\operatorname{coker} D & \simeq \quad \operatorname{coker} d,
\end{tikzcd}$$

so $[\operatorname{coker} \Psi] - [\operatorname{coker} \Psi'] = [\operatorname{coker} \tilde{\psi}] - [\operatorname{coker} \tilde{\psi}']$ in $K_0T(\Lambda(G'_\infty))$, i.e.,

$$\widetilde{\mho}'_S = \operatorname{res}^{G'_\infty}_{G_\infty}(\widetilde{\mho}_S).$$

Remark. (F) If $\widetilde{\Theta}_S$ exists for G_∞, then it exists for all factor groups by a finite normal subgroup and for all open subgroups, provided all of these have $SK_1 = 1$. This follows from Lemma 9 and Proposition 12.

Lemma 13. *$\widetilde{\mho}_S$ is in the image of $\partial : K_1(\mathcal{Q}(G_\infty)) \to K_0T(\Lambda(G_\infty))$.*

Proof. To see this, note that by the localization sequence

$$\widetilde{\mho}_S \in \operatorname{im}(\partial) \quad \Longleftrightarrow \quad [\operatorname{coker} \Psi] = 0 \text{ in } K_0\big(\Lambda(G_\infty)\big);$$

moreover, $[\operatorname{coker}\Psi] = [Y_\infty] - [\Lambda(G_\infty)]$, by definition.

We proceed by induction on the index in G_∞ of the pro-l primary part, Z, of the centre of G_∞. Let G' be a proper subgroup of G_∞ containing Z. Then we have

$$\begin{tikzcd}
K_0T(\Lambda(G_\infty)) \arrow[r] \arrow[d, "\mathrm{res}"'] & K_0(\Lambda(G_\infty)) \arrow[r, tail, "1"] \arrow[d, "\mathrm{res}"'] & K_0(\mathbb{Z}_l[G_\infty/Z]) \arrow[r, tail, "2"] \arrow[d, "\mathrm{res}"'] & K_0(\mathbb{Q}_l[G_\infty/Z]) \arrow[d, "\mathrm{res}"'] \\
K_0T(\Lambda(G')) \arrow[r] & K_0(\Lambda(G')) \arrow[r] & K_0(\mathbb{Z}_l[G'/Z]) \arrow[r] & K_0(\mathbb{Q}_l[G'/Z])
\end{tikzcd}$$

In the diagram, map $\overset{1}{\rightarrowtail}$ is injective by [1, I, 6.17, p. 130] and map $\overset{2}{\rightarrowtail}$ by Swan's theorem [1, I, 32.1, p. 671].

Starting with $\widetilde{\mho}_S$ at the top left, the induction hypothesis implies that its image in $K_0(\Lambda(G'))$ is 0, by Proposition 12. So the image x of $\widetilde{\mho}_S$ in $K_0(\mathbb{Q}_l[G_\infty/Z])$ restricts to zero for every proper subgroup of G_∞/Z. If G_∞/Z is not cyclic, this implies that the character x of the group G_∞/Z vanishes and the diagram finishes the proof. If G_∞/Z is cyclic, then G_∞ is Abelian and the result is [12, Lemma 5c]. □

Proposition 14. *Let $\chi \in \mathrm{Irr}_l(G_\infty)$. The image of $\widetilde{\mho}_S$ under the map $K_0T(\Lambda(G_\infty)) \to K_0T(\Lambda^{\mathfrak{o}}(\Gamma_k))$ of* (2) *of Proposition* 10 *is*

$$\left[\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} X_\infty)\right] - \left[\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} \mathbb{Z}_l)\right].$$

Proof. For the proof recall

$$\begin{tikzcd}
 & \Lambda(G_\infty) \arrow[r, equal] \arrow[d, tail, "\Psi"'] & \Lambda(G_\infty) \arrow[d, tail, "\psi"'] \\
X_\infty \arrow[r, tail] & Y_\infty \arrow[r, two heads] & \Delta G_\infty
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Lambda(G_\infty) \arrow[r, equal] \arrow[d, tail, "\psi"'] & \Lambda(G_\infty) \arrow[d, tail, "\tilde{\psi}"'] & \\
\Delta G_\infty \arrow[r, tail] & \Lambda(G_\infty) \arrow[r, two heads] & \mathbb{Z}_l
\end{tikzcd}$$

Looking at cokernels we obtain exact sequences

$$X_\infty \rightarrowtail \operatorname{coker}\Psi \twoheadrightarrow \operatorname{coker}\psi \quad \text{and} \quad \operatorname{coker}\psi \rightarrowtail \operatorname{coker}\tilde{\psi} \twoheadrightarrow \mathbb{Z}_l$$

of finitely generated $\Lambda(G_\infty)$-modules. Applying $\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} -)$ gives the exact sequences (compare the proof of (2) of Proposition 10)

$$
\begin{aligned}
&\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} X_\infty) \rightarrowtail \mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \operatorname{coker}\Psi)\\
&\quad\to \mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \operatorname{coker}\psi) \twoheadrightarrow H^1\big(H, \mathrm{Hom}_{\mathfrak{o}}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} X_\infty)\big)\\
&\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \operatorname{coker}\psi) \rightarrowtail \mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \operatorname{coker}\tilde{\psi})\\
&\quad\to \mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \mathbb{Z}_l) \twoheadrightarrow H^1\big(H, \mathrm{Hom}_{\mathfrak{o}}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \operatorname{coker}\psi)\big)
\end{aligned}
$$

because $\operatorname{coker}\Psi$ and $\operatorname{coker}\tilde{\psi}$ have finite projective dimension. They in turn imply that the image of $\widetilde{\mho}_S$ is

$$
\begin{aligned}
&\big[\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \operatorname{coker}\Psi)\big] - \big[\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \operatorname{coker}\tilde{\psi})\big]\\
&\quad= \big[\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} X_\infty)\big] - \big[H^1\big(H, \mathrm{Hom}_{\mathfrak{o}}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} X_\infty)\big)\big]\\
&\qquad- \big[\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \mathbb{Z}_l)\big] + \big[H^1\big(H, \mathrm{Hom}_{\mathfrak{o}}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \operatorname{coker}\psi)\big)\big].
\end{aligned}
$$

We are left with showing that the H^1-terms vanish. This is done by dimension shift in Tate cohomology. The exact sequence $X_\infty \rightarrowtail Y_\infty \to \Lambda(G_\infty) \twoheadrightarrow \mathbb{Z}_l$ induces

$$
H^1\big(H, \mathrm{Hom}_{\mathfrak{o}}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} X_\infty)\big) \simeq H^{-1}\big(H, \mathrm{Hom}_{\mathfrak{o}}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \mathbb{Z}_l)\big)
$$

since $\mathrm{Hom}_{\mathfrak{o}}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} Y_\infty)$ and $\mathrm{Hom}_{\mathfrak{o}}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \Lambda(G_\infty))$ are H-cohomologically trivial. The group H^{-1} is finite, since it is annihilated by $|H|$ and since $\mathrm{Hom}_{\mathfrak{o}}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \mathbb{Z}_l)$ is finitely generated over $\mathfrak{o}$. This already takes care of the second term above, because of [12, Proposition 4], by which finite modules are trivial in K_0T. With respect to the last term, we do almost the same but with $X_\infty \rightarrowtail Y_\infty \to \Lambda(G_\infty) \twoheadrightarrow \mathbb{Z}_l$ replaced by the two short exact sequences $\Lambda(G_\infty) \rightarrowtail \Delta G_\infty \twoheadrightarrow \operatorname{coker}\psi$ and $\Delta G_\infty \rightarrowtail \Lambda(G_\infty) \twoheadrightarrow \mathbb{Z}_l$ yielding

$$
\begin{aligned}
H^1\big(H, \mathrm{Hom}_{\mathfrak{o}}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \operatorname{coker}\psi)\big) &\simeq H^1\big(H, \mathrm{Hom}_{\mathfrak{o}}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \Delta G_\infty)\big)\\
&\simeq \hat{H}^0\big(H, \mathrm{Hom}_{\mathfrak{o}}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \mathbb{Z}_l)\big). \qquad \square
\end{aligned}
$$

Lemma 15. *Given a topological generator γ_k of Γ_k, the element in $K_1(\mathcal{Q}^F(\Gamma_k))$ having determinant $H_\chi(\gamma_k - 1)$ is sent to* $[\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o}\otimes_{\mathbb{Z}_l} \mathbb{Z}_l)]$ *by* $K_1(\mathcal{Q}^F(\Gamma_k)) \xrightarrow{\partial} K_0T(\Lambda^{\mathfrak{o}}(\Gamma_k))$.

Proof. To see this, we employ the diagram shown in (3) of Proposition 10,

$$
\begin{tikzcd}
K_1(\mathcal{Q}(G_\infty)) \arrow[r, "\partial"] \arrow[d] & K_0T(\Lambda(G_\infty)) \arrow[d] \\
K_1(\mathcal{Q}^F(\Gamma_k)) \arrow[r, "\partial"] \arrow[d, "\det"'] & K_0T(\Lambda^{\mathfrak{o}}(\Gamma_k)) \\
\mathcal{Q}^F(\Gamma_k)^\times &
\end{tikzcd}
$$

and also Remark (A) in Section 4, by which $[\mathcal{Q}(G_\infty), (\gamma - 1)e + (1-e)] \in K_1(\mathcal{Q}(G_\infty))$ has image $H_\chi(\gamma_k - 1)$ in $\mathcal{Q}^F(\Gamma_k)^\times$, whence the image in $K_0T(\Lambda^{\mathfrak{o}}(\Gamma_k))$ is $[\Lambda^{\mathfrak{o}}(\Gamma_k)/\Lambda^{\mathfrak{o}}(\Gamma_k)\cdot H_\chi(\gamma_k - 1)]$, by Lemma 3.

Now, if χ is not type W, $\Lambda^{\mathfrak{o}}(\Gamma_k) = \Lambda^{\mathfrak{o}}(\Gamma_k) \cdot H_\chi(\gamma_k - 1)$ and $\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} \mathbb{Z}_l) = 0$. On the other hand, if χ is type W, so $H_\chi(\gamma_k - 1) = \chi(\gamma_k)\gamma_k - 1$, then $\Lambda^{\mathfrak{o}}(\Gamma_k)/\Lambda^{\mathfrak{o}}(\Gamma_k) \cdot H_\chi(\gamma_k - 1) \simeq \mathfrak{o}$ with γ_k acting by $\chi(\gamma_k)^{-1}$, and choosing $M_\chi = \mathfrak{o}$ (with γ_k acting by $\chi(\gamma_k)$) yields $\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} \mathbb{Z}_l) = \mathrm{Hom}_{\mathfrak{o}[H]}(\mathfrak{o}, \mathfrak{o}) \simeq \mathfrak{o}$, with γ_k acting by $\chi(\gamma_k)^{-1}$. $\square$

Writing ${\mathbb{Z}_l}^{\mathrm{c}}$ for the integral closure of $\mathbb{Z}_l$ in ${\mathbb{Q}_l}^{\mathrm{c}}$, we set $\Lambda^{\mathrm{c}}(\Gamma_k) = {\mathbb{Z}_l}^{\mathrm{c}} \otimes_{\mathbb{Z}_l} \Lambda(\Gamma_k)$.

Theorem 16. *If* $x \in K_1(\mathcal{Q}(G_\infty))$ *has* $\partial(x) = \widetilde{\mho}_S$, *then* $\mathrm{Det}(x) L_k^{-1} \in \mathrm{Hom}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}(\Gamma_k)^\times)$.

Proof. We first show that it suffices to assume that G_∞ is Abelian. Let χ be a fixed character of G_∞ (with open kernel) and write $\chi = \sum_i n_i \, \mathrm{ind}_{G_i'}^{G_\infty}(\chi_i')$, by Brauer induction, with open subgroups $G_i' \leqslant G_\infty$ and degree 1 characters χ_i' of G_i'. Moreover, let k_i, K_i be the fixed fields of G_i' and its commutator subgroup, respectively, and write $\chi_i' = \mathrm{infl}_{(G_i')^{\mathrm{ab}}}^{G_i'} \chi_i$ (note $[K_\infty : K_i] < \infty$). Then

$$x_i \overset{\mathrm{def}}{=} \mathrm{defl}_{G_i'}^{(G_i')^{\mathrm{ab}}} \mathrm{res}_{G_\infty}^{G_i'}(x) \in K_1\big(\mathcal{Q}((G_i')^{\mathrm{ab}})\big) \text{ satisfies } \partial(x_i) = \mho_S\widetilde{(K_i/k_i)}$$

in $K_0 T(\Lambda((G_i')^{\mathrm{ab}}))$ by Proposition 12. We have

$$\begin{aligned}
&\big(\mathrm{Det}(x) \cdot L_k^{-1}\big)(\chi) \\
&= \prod_i \big[(\mathrm{Det}\, x)\big(\mathrm{ind}_{G_i'}^{G_\infty} \mathrm{infl}_{(G_i')^{\mathrm{ab}}}^{G_i'} \chi_i\big) \cdot L_k\big(\mathrm{ind}_{G_i'}^{G_\infty} \mathrm{infl}_{(G_i')^{\mathrm{ab}}}^{G_i'} \chi_i\big)^{-1}\big]^{n_i} \\
&= \prod_i \big[\big(\mathrm{defl}_{G_i'}^{(G_i')^{\mathrm{ab}}} \mathrm{res}_{G_\infty}^{G_i'} \mathrm{Det}(x)\big)(\chi_i) \cdot \big(\mathrm{defl}_{G_i'}^{(G_i')^{\mathrm{ab}}} \mathrm{res}_{G_\infty}^{G_i'} L_k\big)(\chi_i)^{-1}\big]^{n_i} \\
&= \prod_i \big[(\mathrm{Det}\, x_i)(\chi_i) \cdot L_{k_i}(\chi_i)^{-1}\big]^{n_i},
\end{aligned}$$

as required by K_i/k_i Abelian.

From now on, G_∞ is Abelian and χ an irreducible character. We may suppose that $\mathfrak{o} = \mathbb{Z}_l[\chi]$. We combine the diagram in (3) of Proposition 10 with Proposition 14,

$$\begin{array}{ccccccc}
x & \in & K_1(\mathcal{Q}(G_\infty)) & \xrightarrow{\partial} & K_0 T(\Lambda(G_\infty)) & \ni & \widetilde{\mho}_S \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
x_\chi & \in & K_1(\mathcal{Q}^F(\Gamma_k)) & \xrightarrow{\partial} & K_0 T(\Lambda^{\mathfrak{o}}(\Gamma_k)) & \ni & [\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} X_\infty)] - [\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} \mathbb{Z}_l)] \\
 & & \det \downarrow & & & & \\
 & & \mathcal{Q}^F(\Gamma_k)^\times & & & &
\end{array}$$

and deduce from $(\mathrm{Det}\, x)(\chi) = \det(x_\chi)$ that it remains to show

$$(*) \qquad \det(x_\chi) L_k(\chi)^{-1} \in \Lambda^{\mathfrak{o}}(\Gamma_k)^\times.$$

Choose a topological generator γ_k of Γ_k and a preimage $\gamma \in G_\infty$. Twist the character χ by a type W character ρ so that $\chi(\gamma) = 1$ results, i.e., χ is now of type S with respect to γ. Since $\mathrm{Det}(x)$ and L_k belong to $\mathrm{Hom}^*(R_l(G_\infty), \mathcal{Q}^c(\Gamma_k)^\times)$, the twisting does not effect statement $(*)$ because automorphisms $\rho^\sharp$ of $\mathcal{Q}^c(\Gamma_k)$ preserve $\Lambda^c(\Gamma_k)$. By Lemma 15 we are left to show

$$(**) \qquad \det(\bar{x}_\chi) G_{\chi,S}(\gamma_k - 1)^{-1} \in \Lambda^{\mathfrak{o}}(\Gamma_k)^\times$$

with $\partial(\bar{x}_\chi) = [\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} X_\infty)]$, for χ with $\chi(\gamma) = 1$.

From $\chi(\gamma) = 1$ we get (as in the proof of Lemma 2)

$$\mathrm{Hom}_{F[H]}(V_\chi, F \otimes_{\mathbb{Z}_l} X_\infty) = (F \otimes_{\mathbb{Z}_l} X_\infty)^{(\chi)}$$

the subspace of $F \otimes_{\mathbb{Z}_l} X_\infty$ on which H acts via χ (as in [18, p. 495]). Thus (compare Lemma 3) $\mathrm{char}([\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} X_\infty)])$ is $G^*_{\chi,S}(T)$ by [18, p. 496].

This proves the distinguished polynomial part of $(**)$.

It remains to show that $\mu_{\mathfrak{o}}([\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} X_\infty)])$ is the exponent of λ in the Weierstraß decomposition of $G_{\chi,S}(\gamma_k - 1)$. This follows from [11, Section 5] but can also be seen more directly: we choose $M_\chi = \mathfrak{o}$ with the G_∞-action given by χ. Then

$$\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} X_\infty) = X_\infty^\chi = \left\{x \in \mathfrak{o} \otimes_{\mathbb{Z}_l} X_\infty \colon hx = \chi(h)x \text{ for all } h \in H\right\},$$

now in the notation of [11, p. 497].

The Galois group $G_{\mathbb{Q}_l^c/\mathbb{Q}_l}$ acts on $\Lambda^{\mathfrak{o}}(\Gamma_k) = \mathfrak{o} \otimes_{\mathbb{Z}_l} \Lambda(\Gamma_k)$ via $\mathfrak{o}$. The induced action on $K_0T(\Lambda^{\mathfrak{o}}(\Gamma_k))$ takes $[X_\infty^\chi]$ to $[X_\infty^{\chi^\sigma}]$ (with $\sigma \in G_{\mathbb{Q}_l^c/\mathbb{Q}_l}$) and is trivial on $K_0T(\Lambda^{\mathfrak{o}}(\Gamma_k)_\bullet)$, because $\Lambda^{\mathfrak{o}}(\Gamma_k)_\bullet$ has only one simple module. Thus $\mu_{\mathfrak{o}}(X_\infty^\chi) = \mu_{\mathfrak{o}}(X_\infty^{\chi^\sigma})$ by (3) of Lemma 3.

Decompose $\chi = \alpha\beta$ into the product of characters α, β of l-power order, respectively of order prime to l. Let $A = \ker\alpha$ and let A' be the group between A and G_∞ with $[A' : A] = l$. Then

$$(***) \qquad \beta \cdot \mathrm{ind}_A^{G_\infty}(1) - \beta \cdot \mathrm{ind}_{A'}^{G_\infty}(1) = \sum_{\sigma \in G_{\mathbb{Q}_l(\chi)/\mathbb{Q}_l(\beta)}} \chi^\sigma.$$

When G' is a subgroup of G_∞ containing γ and χ' is a character of G' with $\chi'(\gamma) = 1$, we also write $M_{\chi'} = \mathfrak{o}$ as before, i.e., G' acts via χ'. Now,

$$\left[\mathrm{Hom}_{\mathfrak{o}[H]}\left(\mathrm{ind}_A^{G_\infty}(M_{\mathrm{res}^A_{G_\infty}\beta}), \mathfrak{o} \otimes_{\mathbb{Z}_l} X_\infty\right)\right] = \left[\mathrm{Hom}_{\mathfrak{o}[H\cap A]}(M_{\mathrm{res}^A_{G_\infty}\beta}, \mathfrak{o} \otimes_{\mathbb{Z}_l} X_\infty)\right]$$

in $K_0T(\Lambda^{\mathfrak{o}}(\Gamma_k))$ by Frobenius reciprocity, because γ acts trivially. Since $\mathrm{ind}_A^{G_\infty}(M_{\mathrm{res}^A_{G_\infty}\beta})$ and $\mathrm{ind}_{A'}^{G_\infty}(M_{\mathrm{res}^{A'}_{G_\infty}\beta}) \oplus \bigoplus_\sigma M_\chi^\sigma$ have the same character by $(***)$, (2) of Proposition 10 now implies

$$\mu_{\mathfrak{o}}\left([X_\infty^{\mathrm{res}^A_{G_\infty}\beta}]\right) - \mu_{\mathfrak{o}}\left([X_\infty^{\mathrm{res}^{A'}_{G_\infty}\beta}]\right) = \left[\mathbb{Q}_l(\chi) : \mathbb{Q}_l(\beta)\right] \cdot \mu_{\mathfrak{o}}\left([X_\infty^\chi]\right).$$

Similarly, from properties (4) and (5) in Section 4 (with $\phi_\chi = G_{\chi,S}$),

$$\begin{aligned}&\mu_{\mathfrak{o}}\big(G_{\mathrm{res}^{A}_{G_\infty}\beta,S}(\gamma_k - 1)\big) - \mu_{\mathfrak{o}}\big(G_{\mathrm{res}^{A'}_{G_\infty}\beta,S}(\gamma_k - 1)\big)\\ &\quad = \big[\mathbb{Q}_l(\chi) : \mathbb{Q}_l(\beta)\big] \cdot \mu_{\mathfrak{o}}\big(G_{\chi,S}(\gamma_k - 1)\big).\end{aligned}$$

By [18, Theorem 1.4] the left-hand sides agree on replacing $\mathfrak{o}$ by $\mathbb{Z}_l[\beta]$. Dividing by $[\mathbb{Q}_l(\chi) : \mathbb{Q}_l(\beta)]$ changes $\mu_{\mathfrak{o}}$ into $\mu_{\mathbb{Z}_l[\beta]}$ and completes the proof. □

Remarks. (G)

$$G_{\chi,S}(T) \in \mathbb{Z}_l[\chi][[T]] \quad \text{for all } \chi \in \mathrm{Irr}_l(G_\infty).$$

This follows from $[\mathcal{Q}^F(\Gamma_k), G_{\chi,S}(\gamma_k - 1)] \mapsto [\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} X_\infty)]$ by ∂: $K_1(\mathcal{Q}^F(\Gamma_k)) \to K_0T(\Lambda^{\mathfrak{o}}(\Gamma_k))$. Hence $G_{\chi,S}(T)$ is the K_1-characteristic polynomial of the torsion $\Lambda^{\mathfrak{o}}(\Gamma_k)$-module $\mathrm{Hom}_{\mathfrak{o}[H]}(M_\chi, \mathfrak{o} \otimes_{\mathbb{Z}_l} X_\infty)$ and thus integral. Note that the statement is independent of S (containing l and ∞) because Euler factors of $G_{\chi,S}(\gamma_k - 1)$ and changes in $\mho_S$ both have trivial image in $K_0T(\Lambda^{\mathfrak{o}}(\Gamma_k)_\bullet)$ (for the latter see ((4.∗)) on p. 37 of [11]).

(H) If $\widetilde{\Theta}_S$ exists, then $x = \widetilde{\Theta}_S \Upsilon$ with $\mathrm{nr}(\Upsilon) \in \mathfrak{z}^\times$, where $\mathfrak{z}$ is the integral closure of $\mathrm{cent}(\Lambda(G_\infty))$ in $\mathrm{cent}(\mathcal{Q}(G_\infty))$.

This follows from Theorem 16 because $\mathfrak{z}$ is a maximal $\Lambda(\Gamma)$-order in $\mathrm{cent}(\Lambda(G_\infty))$ (with Γ, as before, a central open subgroup of G_∞) and so the isomorphism (∗) in the proof of Theorem 8 induces $\mathfrak{z} \simeq \mathrm{Maps}^{\mathrm{W}}_{G(\mathbb{Q}_l{}^c/\mathbb{Q}_l)}(\mathrm{Irr}_l(G_\infty), \Lambda^{\mathrm{c}}(\Gamma_k))$. Note that "$x = \widetilde{\Theta}_S \Upsilon$ with $\mathrm{nr}(\Upsilon) \in \mathfrak{z}^\times$" is the analogue of [12, Theorem 6(a)] and thus may be viewed as a "maximal order variant" of conjecture ($\mathfrak{F}$).

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